The Converse of the Real Orthogonal Holant Theorem and Symmetric Tensor Diagonalization

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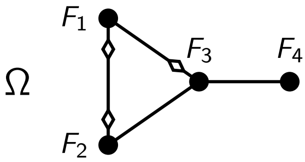
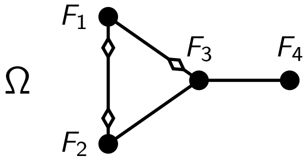
Signature Grids

- **Signature** $F : [q]^n \to \mathbb{R}$
  - **Domain** $[q] := \{0, 1, \ldots, q-1\}$
  - **Arity** $n \geq 0$
- e.g. $q = 2$, $n = 3$: $F(x_1, x_2, x_3)$ for Boolean variables x_1, x_2, x_3.
- Let $\mathcal{F}$ be a set of signatures.
- $\mathcal{F}$-**grid** Ω is a multigraph with a signature from $\mathcal{F}$ on each vertex
  - Arity of signature equals degree of vertex

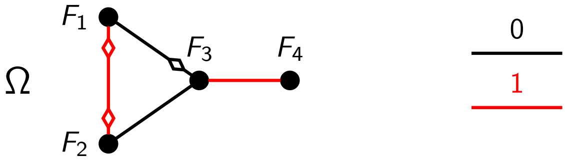


Holant Problems



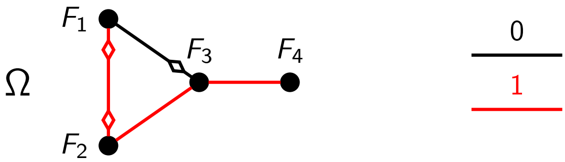
- Let F_v be the function on vertex v.
- Let $\delta(v)$ be an ordered list of edges incident to v.
- Goal: compute the **Holant value** of Ω:

$$\text{Holant}_{\mathcal{F}}(\Omega) = \sum_{\sigma : E(\Omega) \to [q]} \prod_{v \in V(\Omega)} F_v(\sigma(\delta(v))).$$



Example: domain $q = 2$:

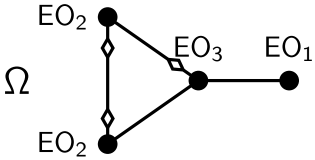
$$\text{Holant}_{\mathcal{F}}(\Omega) = F_1(1,0) \cdot F_2(1,0) \cdot F_3(0,0,1) \cdot F_4(1) +$$



Example: domain $q = 2$:

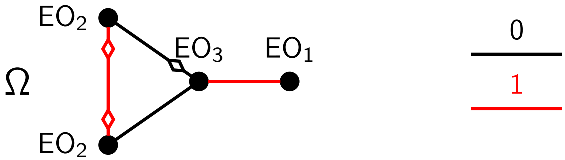
$$\begin{aligned}\text{Holant}_{\mathcal{F}}(\Omega) = \; & F_1(1,0) \cdot F_2(1,0) \cdot F_3(0,0,1) \cdot F_4(1) + \\ & F_1(1,0) \cdot F_2(1,1) \cdot F_3(0,1,1) \cdot F_4(1) + \\ & \ldots\end{aligned}$$

Example: Counting Perfect Matchings



- $EO_n : \{0,1\}^n \to \{0,1\}$ – **ExactOne** signature.
- $EO_n(x_1, \ldots, x_n) = 1$ iff exactly one $x_i = 1$.

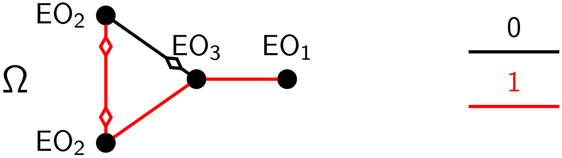
Example: Counting Perfect Matchings



$$\text{Holant}_{EO}(\Omega) = EO_1(1,0) \cdot EO_2(1,0) \cdot EO_3(0,0,1) \cdot EO_4(1) +$$

$$= 1 \cdot 1 \cdot 1 \cdot 1 +$$

Example: Counting Perfect Matchings



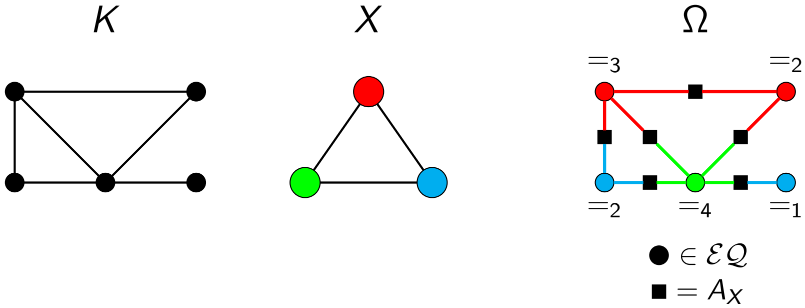
$$
\begin{aligned}
\text{Holant}_{\text{EO}}(\Omega) = \ & \text{EO}_1(1,0)\cdot\text{EO}_2(1,0)\cdot\text{EO}_3(0,0,1)\cdot\text{EO}_4(1) + \\
& \text{EO}_1(1,0)\cdot\text{EO}_2(1,{\color{red}1})\cdot\text{EO}_3(0,{\color{red}1},1)\cdot\text{EO}_4(1) + \\
& \dots \\
= \ & 1\cdot 1\cdot 1\cdot 1 + \\
& 1\cdot 0\cdot 0\cdot 1 + \\
& \dots
\end{aligned}
$$

Example: Counting Graph Homomorphisms

- $\phi : V(K) \to V(X)$ is a **graph homomorphism** if it maps all edges to edges: $\{u, v\} \in E(K) \implies \{\phi(u), \phi(v)\} \in E(X)$.
- $\mathcal{EQ} = \{=_n \mid n \in \mathbb{N}\}$ where

$$(=_n)(x_1, \ldots, x_n) = \begin{cases} 1 & x_1 = \ldots = x_n \\ 0 & \text{otherwise} \end{cases}$$

- $(\#\text{ homomorphisms } K \to X) = \text{Holant}_{\{A_X\} \cup \mathcal{EQ}}(\Omega)$:
  - Domain $[q] = V(X)$. Here $q = 3$.



Why study Holant?

- Very expressive framework for counting problems.
- But restricted enough to admit **complexity dichotomy theorems**:
- For any signature set $\mathcal{F}$ (of a certain class), $\text{Holant}_{\mathcal{F}}$ is always either in P or #P-hard, with nothing in between.

Broad dichotomies exist on Boolean domain ($q = 2$) for $\mathcal{F}$ containing signatures that are

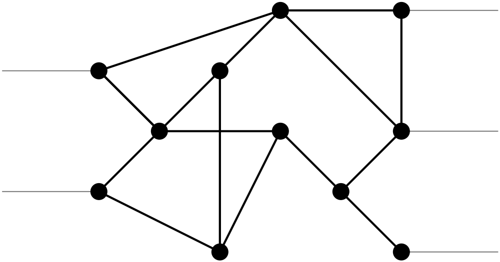
- Complex-valued and symmetric (Cai, Guo, and Williams [CGW16])
- Real-valued (Shao and Cai [SC20])

Weaker dichotomies exist on higher domain $q > 2$ for Holant^* problems:

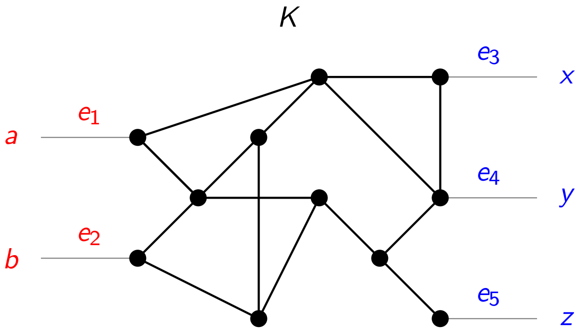
- $q = 3$ and a single complex-valued symmetric ternary function (Cai, Lu, and Xia [CLX13])
- $q = 4$ and a single $\{0, 1\}$-valued symmetric ternary function (Liu, Fan, and Cai [LFC23])

Gadgets

- A **gadget** is a signature grid with **dangling edges**.
- Several signatures assembled into a new signature.
- Inputs along dangling edges.



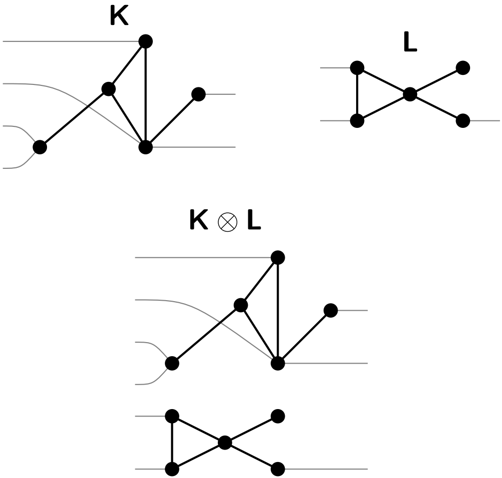
Gadgets and signature matrices



- $a, b, x, y, z \in [q]$
- $q^2 \times q^3$ signature matrix $M(\mathbf{K})$.

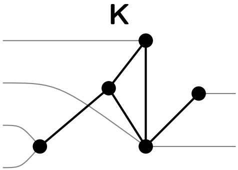
$$M(\mathbf{K})_{ab,xyz} = \sum_{\substack{\sigma: E(\mathbf{K}) \to [q] \\ \sigma(e_1,e_2)=(a,b) \\ \sigma(e_3,e_4,e_5)=(x,y,z)}} \prod_{v \in V(\mathbf{K})} F_v(\sigma(\delta(v))).$$

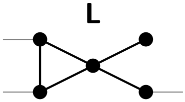
Gadget operations

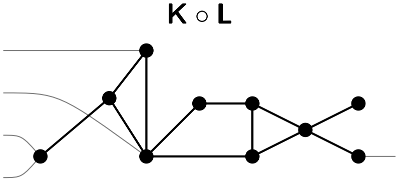


$$M(\mathbf{K} \otimes \mathbf{L}) = M(\mathbf{K}) \otimes M(\mathbf{L})$$

Gadget operations

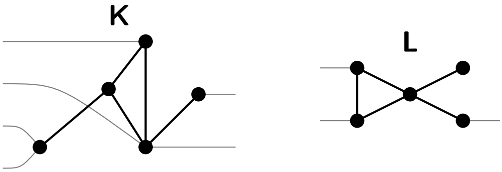


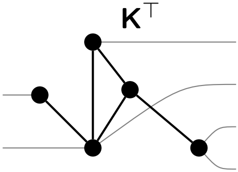




$$M(\mathbf{K} \circ \mathbf{L}) = M(\mathbf{K}) \circ M(\mathbf{L})$$

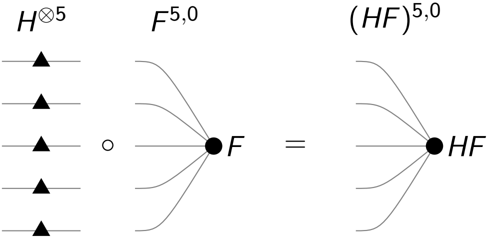
Gadget operations





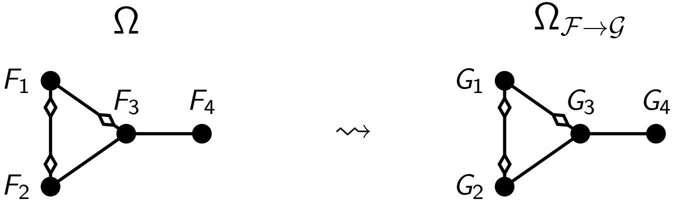
$$M(\mathbf{K}^\top) = M(\mathbf{K})^\top$$

- Let H be $q \times q$ matrix, $F : [q]^n \to \mathbb{R}$.
- Define $HF : [q]^n \to \mathbb{R}$ by applying H to each input of F:



- $H^{\otimes n}$ is $q^n \times q^n$ matrix, $F^{n,0}$ is length-q^n vector.
- HF is F under basis H.
- For signature set $\mathcal{F}$, define $H\mathcal{F} := \{HF \mid F \in \mathcal{F}\}$.

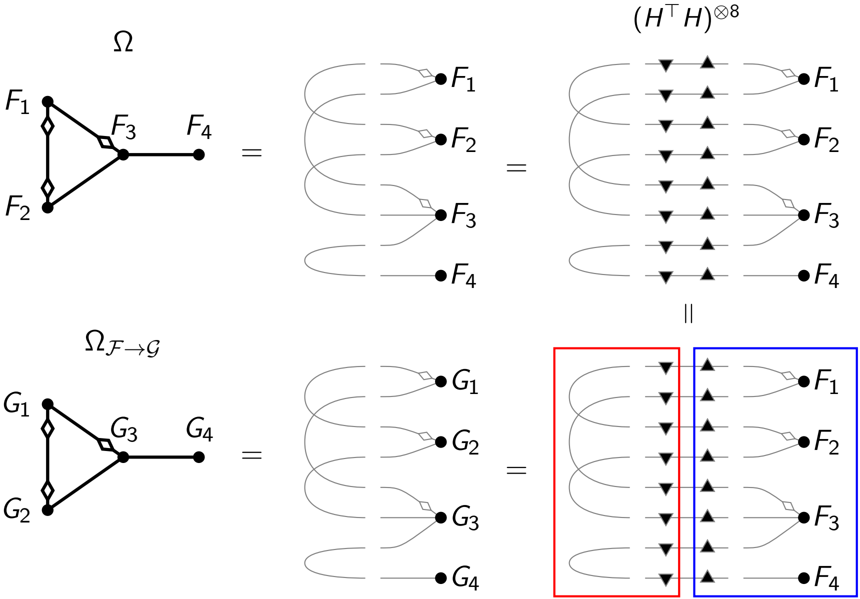
- Let $\mathcal{F}$ and $\mathcal{G}$ be signature sets on same domain $[q]$.
- Assume there is a bijection between $\mathcal{F}$ and $\mathcal{G}$ preserving arity.
- For $\mathcal{F}$-grid Ω, define $\mathcal{G}$-grid $\Omega_{\mathcal{F}\to\mathcal{G}}$ by replacing every $F \in \mathcal{F}$ by the corresponding $G \in \mathcal{G}$.

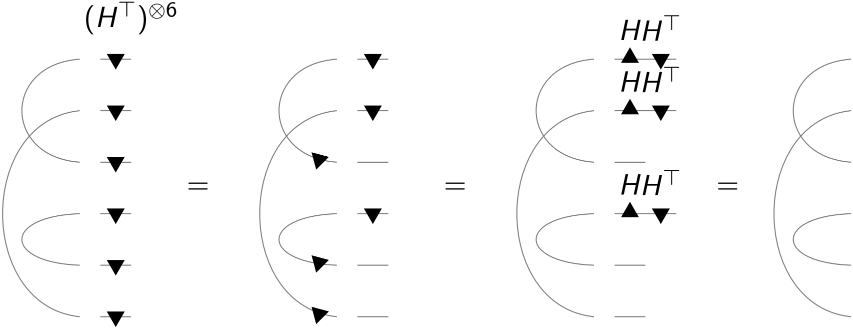


Theorem (The Orthogonal Holant Theorem)

If $\mathcal{G} = H\,\mathcal{F}$ for some orthogonal H, then, for every $\mathcal{F}$-grid Ω,

$$\text{Holant}_{\mathcal{F}}(\Omega) = \text{Holant}_{\mathcal{G}}(\Omega_{\mathcal{F}\to\mathcal{G}}).$$


Ω
$(H^\top H)^{\otimes 8}$
F_1
F_2
F_3
F_4
$=$
$\Omega_{\mathcal{F}\to\mathcal{G}}$
G_1
G_2
G_3
G_4


$(H^\top)^{\otimes 6}$
$=$
$=$
$HH^\top$
$HH^\top$
$HH^\top$
$=$

The Holant Theorem

- The orthogonal Holant theorem is a special case of Valiant's general Holant theorem. [Val08]
- *Holographic algorithms* using the Holant theorem are the original motivation for Holant problems.
- Xia conjectured the converse of the Holant theorem [Xia10].
- Converse does not hold in general [CGW16]
- But we show it does in the orthogonal case:

Definition

$\mathcal{F}$ and $\mathcal{G}$ are **Holant-indistinguishable** if, for every $\mathcal{F}$-grid Ω,

$$\text{Holant}_{\mathcal{F}}(\Omega) = \text{Holant}_{\mathcal{G}}(\Omega_{\mathcal{F}\to\mathcal{G}}).$$

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for orthogonal H if and only if $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for some orthogonal H if and only if $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

- This is a **counting indistinguishability theorem**
- Two objects are equivalent up to some algebraic transformation iff they are indistinguishable parameters for a counting problem.

Counting Indistinguishability Theorems

- Let X and Y be graphs:

Theorem (Lovász [Lov67])

X and Y are isomorphic iff X and Y are homomorphism-indistinguishable.

Theorem

$HA_X = A_Y H$ for some orthogonal H iff X and Y are homomorphism-indistinguishable over all cycles.

Theorem (Mančinska-Roberson [MR20])

X and Y are quantum isomorphic iff X and Y are homomorphism-indistinguishable over all planar graphs.

- The first two are direct consequences of our main result.

Counting Indistinguishability Theorems

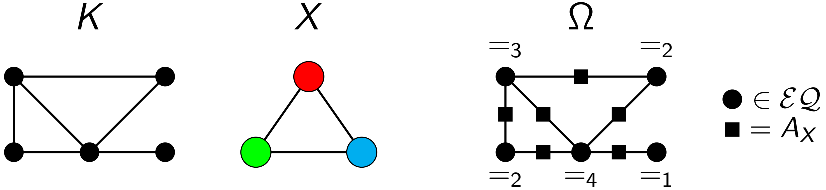
Theorem (Lovász [Lov67])

X and Y are isomorphic iff X and Y are homomorphism-indistinguishable.

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for some orthogonal H iff $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

- Recall: $\#\text{hom}(K, X) \equiv \text{Holant}_{\{A_X\} \cup \mathcal{EQ}}$.



- $H\mathcal{EQ} = \mathcal{EQ}$ iff H is a permutation matrix.
- $H(\{A_X\} \cup \mathcal{EQ}) = \{A_Y\} \cup \mathcal{EQ}$ iff H is a permutation matrix and transforms A_X to A_Y.

Proof of the Converse

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for some orthogonal H iff $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

- Assume $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.
- Frequent idea: Can add new signatures to $\mathcal{F}$ and $\mathcal{G}$ if Holant-indistinguishability is preserved.
- Can assume $\mathcal{F}$ and $\mathcal{G}$ are *gadget-closed*
  - i.e. $\mathcal{F}$ contains all signatures of $\mathcal{F}$-gadgets.
  - Adding gadget signatures preserves Holant-indistinguishability.
- Proof by induction on q (the domain size).
- Assume theorem holds for all $\mathcal{F}', \mathcal{G}'$ on domain smaller than q.

Proof of the Converse: Inductive Lemma

Lemma (Inductive Lemma)

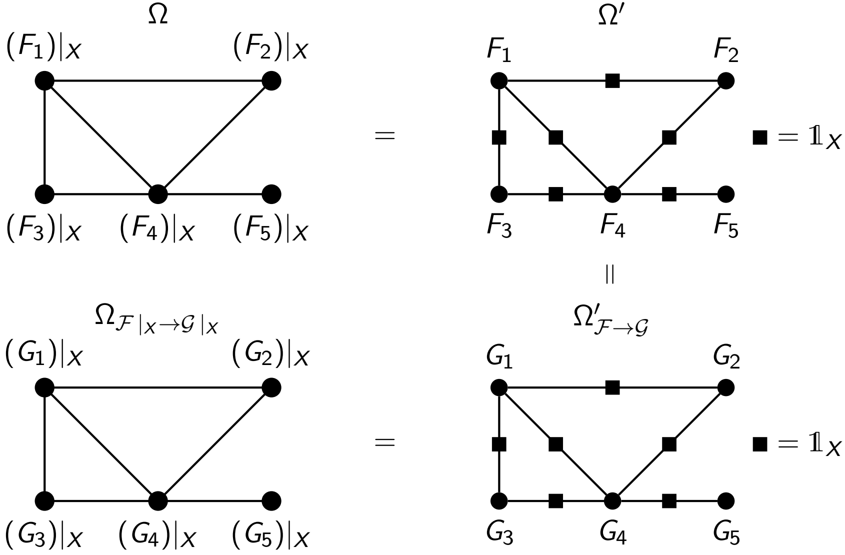
If $\mathcal{F}$ and $\mathcal{G}$ contain a diagonal matrix (binary signature) $D \notin \mathrm{span}(I)$, then there is an orthogonal H such that $\mathcal{G} = H\mathcal{F}$.

$$D = \begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}}_{\mathbb{1}_X}, \underbrace{\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{\mathbb{1}_Y}$$

- Define subdomains $X = \{0, 1, 2\}$, $Y = \{3, 4\} \subset [5] = [q]$
- Interpolate $\mathbb{1}_X, \mathbb{1}_Y \in \mathcal{F}, \mathcal{G}$.
- $\mathcal{F}|_X, \mathcal{G}|_X$: subsignatures on domain X

Proof of the Converse: Inductive Lemma

- $\mathcal{F}|_X$ and $\mathcal{G}|_X$ are Holant-indistinguishable:



Proof of the Converse: Inductive Lemma

- $\mathcal{F}|_X$ and $\mathcal{G}|_X$ are Holant-indistinguishable.
- $\mathcal{F}|_Y$ and $\mathcal{G}|_Y$ are Holant-indistinguishable, similarly.
- $|X|, |Y| < q$.
- So by induction, there are orthogonal H_X, H_Y such that $\mathcal{G}|_X = H_X \mathcal{F}|_X$ and $\mathcal{G}|_Y = H_Y \mathcal{F}|_Y$.
- Combine these into a full transformation H such that $\mathcal{G} = H\mathcal{F}$.
  - Requires some more work...

Lemma (Inductive Lemma)

If $\mathcal{F}$ and $\mathcal{G}$ contain a diagonal matrix (binary signature) $D \notin \mathrm{span}(I)$, then there is an orthogonal H such that $\mathcal{G} = H\mathcal{F}$.

- How to obtain D?

The Stabilizer of the Disjoint Union

- Let F have domain $V(\mathcal{F})$, G have domain $V(\mathcal{G})$, both n-ary
- Define n-ary signature $F \oplus G$ on domain $V(\mathcal{F}) \sqcup V(\mathcal{G})$.
  - Acts as F when all n inputs from $V(\mathcal{F})$.
  - Acts as G when all n inputs from $V(\mathcal{G})$.
  - 0 on mixed inputs from $V(\mathcal{F})$ and $V(\mathcal{G})$.
- $\mathcal{F} \oplus \mathcal{G} := \{F \oplus G \mid \text{ corresponding } F \in \mathcal{F}, G \in \mathcal{G}\}$.

Definition

$\text{Stab}(\mathcal{F} \oplus \mathcal{G}) := \{\text{orthogonal } H \mid H(\mathcal{F} \oplus \mathcal{G}) = \mathcal{F} \oplus \mathcal{G}\}$.

- $H \in \text{Stab}(\mathcal{F} \oplus \mathcal{G})$ indexed by $V(\mathcal{F}) \sqcup V(\mathcal{G})$.

- $H \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ indexed by $V(\mathcal{F}) \sqcup V(\mathcal{G})$.
- H has block form

$$\begin{array}{c} \\ V(\mathcal{F}) \\ V(\mathcal{G}) \end{array} \begin{array}{c} \begin{array}{cc} V(\mathcal{F}) & \qquad V(\mathcal{G}) \end{array} \\ \begin{bmatrix} H_{V(\mathcal{F}),V(\mathcal{F})} & H_{V(\mathcal{F}),V(\mathcal{G})} \\ H_{V(\mathcal{G}),V(\mathcal{F})} & H_{V(\mathcal{G}),V(\mathcal{G})} \end{bmatrix} \end{array}.$$

Lemma (Nonconstructive Lemma)

If $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable, then $\mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ contains an H which is not block-diagonal

- i.e. $H_{V(\mathcal{G}),V(\mathcal{F})} \neq 0$ or $H_{V(\mathcal{G}),V(\mathcal{F})} \neq 0$.
- Proved using invariant-theoretic theorem of Schrijver [Sch08].

Proof of the Converse: A Diagonal Intertwiner

- Suppose WLOG that $H_{V(\mathcal{G}),V(\mathcal{F})} \neq 0$.
- Singular value decomposition: $H_{V(\mathcal{G}),V(\mathcal{F})} = U^\top D V$ with $D \neq 0$.

$$H = \begin{array}{c} \\ V(\mathcal{F}) \\ V(\mathcal{G}) \end{array} \begin{array}{c} \begin{array}{cc} V(\mathcal{F}) & V(\mathcal{G}) \end{array} \\ \begin{bmatrix} * & * \\ U^\top D V & * \end{bmatrix} \end{array}$$

- Apply orthogonal transforms V to $\mathcal{F}$ and U to $\mathcal{G}$.
- $\operatorname{Stab}(\mathcal{F} \oplus \mathcal{G}) \mapsto (V \oplus U) \operatorname{Stab}(\mathcal{F} \oplus \mathcal{G})(V \oplus U)^\top$.

$$H \mapsto \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} * & * \\ U^\top D V & * \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ 0 & U^\top \end{bmatrix} = \begin{bmatrix} * & * \\ D & * \end{bmatrix} \in \operatorname{Stab}(\mathcal{F} \oplus \mathcal{G})$$

- Therefore D *intertwines* $\mathcal{F}$ and $\mathcal{G}$:
- If $F \in \mathcal{F}$ and corresponding $G \in \mathcal{G}$ have arity $2n$, then

$$D^{\otimes n} F^{n,n} = G^{n,n} D^{\otimes n}.$$

Proof of the Converse: A Diagonal Intertwiner

$$D^{\otimes n} F^{n,n} = G^{n,n} D^{\otimes n} \text{ for every corresponding } F \in \mathcal{F}, G \in \mathcal{G}$$

- If $D \in \text{span}(I)$ then $D = \pm I$ so $(\pm I)\mathcal{G} = \mathcal{F}$.
- If $D \notin \text{span}(I)$...

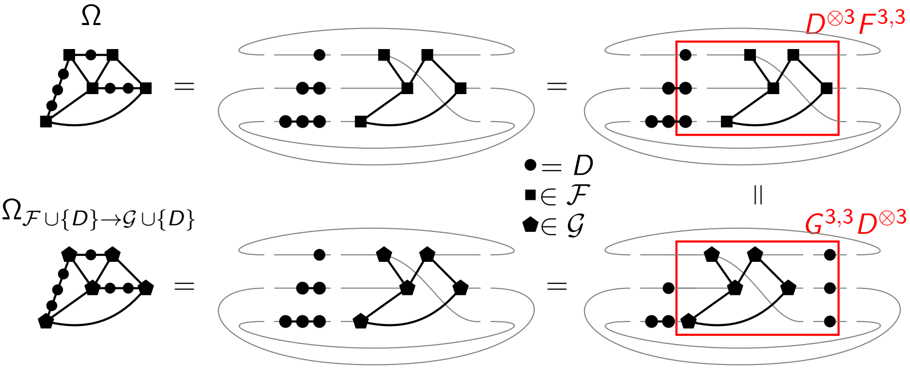
Lemma (Inductive Lemma)

If $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable and contain a diagonal matrix $D \notin \mathit{span}(I)$, then there is an orthogonal H such that $\mathcal{G} = H\mathcal{F}$.

- Show $\mathcal{F} \cup \{D\}$ and $\mathcal{G} \cup \{D\}$ are Holant-indistinguishable (next slide).
- Then Lemma gives orthogonal H such that $(\mathcal{G} \cup \{D\}) = H(\mathcal{F} \cup \{D\})$.
- So $\mathcal{G} = H\mathcal{F}$.

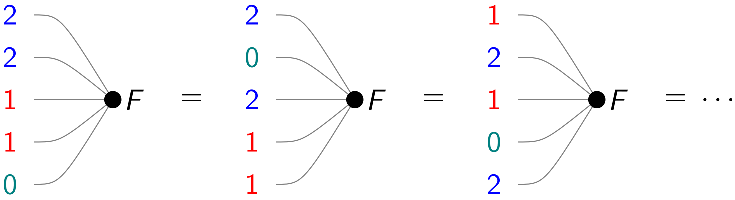
$$D^{\otimes n} F^{n,n} = G^{n,n} D^{\otimes n} \text{ for every corresponding } F \in \mathcal{F}, G \in \mathcal{G}$$

- Recall that $\mathcal{F}$ and $\mathcal{G}$ are gadget-closed.
- Let Ω be an $(\mathcal{F} \cup \{D\})$-grid.

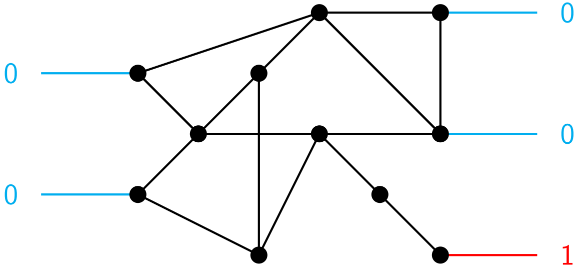


Symmetric Signatures

- F is **symmetric** if its value is invariant under reordering of its inputs



- $E \in \textsc{GenEq}$: $E(x_1, \ldots, x_n) = 0$ unless $x_1 = \ldots = x_n$.
- Every connected $\textsc{GenEq}$-gadget has a signature in $\textsc{GenEq}$:



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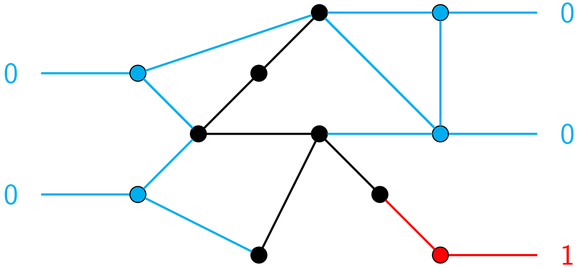

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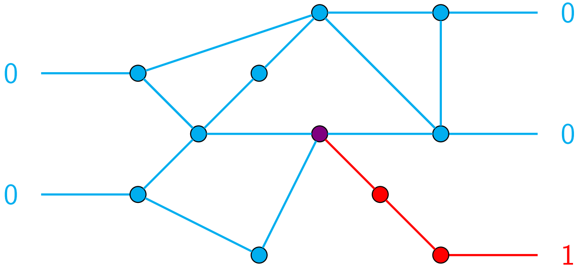


Odeco Signature Sets

- Every connected $\textsc{GenEq}$-gadget has a signature in $\textsc{GenEq}$.
- Every connected $\textsc{GenEq}$-gadget has a symmetric signature.
- $\textsc{GenEq}$ is the **only** signature set with this $\uparrow$ property
  - up to orthogonal transformation.

Definition

$\mathcal{F}$ is **odeco** if $\exists$ orthogonal H such that $H\mathcal{F} \subset \textsc{GenEq}$.

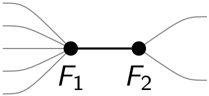
Theorem

$\mathcal{F}$ is odeco iff every connected $\mathcal{F}$-gadget has a symmetric signature.

- In fact, we get something a little stronger...

Odeco Signature Sets

$F_1 * F_2$:



Theorem

The following are equivalent for a set $\mathcal{F}$ of symmetric signatures:

1. $\mathcal{F}$ *is odeco* ($\exists$ *orthogonal* H *s.t.* $H\mathcal{F} \subset \textsc{GenEq}$)
2. *Every connected* $\mathcal{F}$*-gadget has a symmetric signature*
3. $F_1 * F_2$ *is symmetric for every* $F_1, F_2 \in \mathcal{F}$.

- Binary odeco signature $\longleftrightarrow$ diagonalizable matrix.
- If F_1, F_2 are binary, then $F_1 * F_2$ is matrix product.
- $F_1 * F_2$ is symmetric iff F_1 and F_2 commute.
  - $F_1 F_2 = (F_1 F_2)^\top = F_2^\top F_1^\top = F_2 F_1$.
- So 1 $\iff$ 3 says commuting (real, symmetric) matrices are simultaneously diagonalizable.

Odeco Signature Sets

Theorem

The following are equivalent for a set $\mathcal{F}$ of symmetric signatures:

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2. *Every connected* $\mathcal{F}$*-gadget has a symmetric signature*
3. $F_1 * F_2$ *is symmetric for every* $F_1, F_2 \in \mathcal{F}$.

- $1 \iff 3$ extends characterization of [BDHR17].
- $1 \implies 2, 3$:
  - Every $\textsc{GenEq}$ gadget has a symmetric signature,
  - Orthogonal transformation preserves this property
- $3 \implies 2$:
  - Gadget is connected $\implies \exists$ a path between any two dangling edges.
  - Apply symmetry of $F_1 * F_2$ to every vertex along this path.
- $2 \implies 1$: apply the main theorem!

Theorem

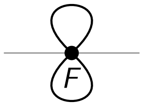
$\mathcal{G} = H\mathcal{F}$ *for orthogonal* H *iff* $\mathcal{F}$ *and* $\mathcal{G}$ *are Holant-indistinguishable.*

Theorem

Every connected $\mathcal{F}$-gadget has a symmetric signature $\implies$ $\mathcal{F}$ is odeco

- Goal: find a $\mathcal{G} \subset \textsc{GenEq}$ s.t. $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.
- Consider $\mathcal{F}$-grid Ω containing signatures $F_1, \ldots, F_p \in \mathcal{F}$.
- Can assume every F_i has even arity
  - Replace F_i with $F_i * F_i$.
- Break an edge of Ω to produce a connected binary gadget $\mathbf{K}$.

$\widetilde{F}$:



Lemma

$$M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F}_i.$$

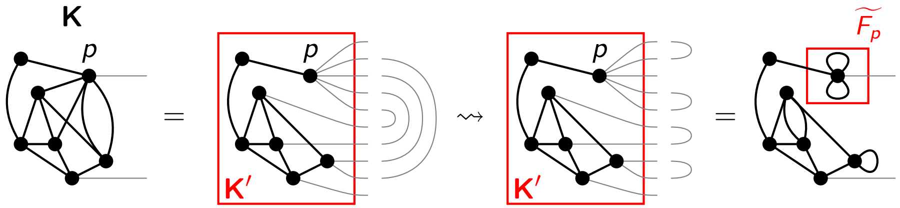
Theorem

Every connected $\mathcal{F}$-gadget has a symmetric signature $\Longrightarrow$ $\mathcal{F}$ is odeco

Lemma

$$M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F}_i.$$

- Induction on p (the number of vertices)



Theorem

Every connected $\mathcal{F}$-gadget has a symmetric signature $\implies$ $\mathcal{F}$ is odeco

Lemma (proved)

$$M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F}_i.$$

- Each $\widetilde{F}_i \widetilde{F}_j$ is symmetric, so $\widetilde{F}_i$ and $\widetilde{F}_j$ commute.
- Thus $\{\widetilde{F}_i\}_{i=1}^{p}$ are simultaneously diagonalizable under basis H.
- Replace $\mathcal{F}$ with $H\,\mathcal{F}$ to assume each $\widetilde{F}_i = \text{diag}(\mathbf{v}^i)$.
- Define $G_i := n_i\text{-ary-diag}(\mathbf{v}^i) \in \textsc{GenEq}$.
  - $n_i := \text{arity}(F_i)$.
  - i.e. $G_i(\underbrace{x, \ldots, x}_{n_i}) = (\mathbf{v}^i)_x$.

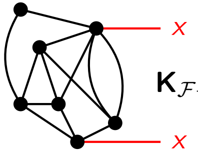
Theorem

Every connected $\mathcal{F}$-gadget has a symmetric signature $\implies$ $\mathcal{F}$ is odeco

Lemma (proved)

$$M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F}_i.$$

- $\widetilde{F}_i = \mathrm{diag}(\mathbf{v}^i)$.
- $G_i \in \textsc{GenEq}$ and $(\mathbf{v}^i)_x = G_i(\underbrace{x, \ldots, x}_{n_i})$.



$$M(\mathbf{K})_{x,x} = \prod_{i=1}^{p} (\mathbf{v}^i)_x = \prod_{i=1}^{p} G_i(x, \ldots, x) = M(\mathbf{K}_{\mathcal{F}\to\mathcal{G}})_{x,x}.$$

- Connect the two dangling edges of $\mathbf{K}$ to recreate Ω.
- $M(\mathbf{K}) = M(\mathbf{K}_{\mathcal{F}\to\mathcal{G}})$, so $\mathrm{Holant}(\Omega) = \mathrm{Holant}(\Omega_{\mathcal{F}\to\mathcal{G}})$.
- Thus $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.
- By main theorem, $\exists$ orthogonal H such that $H\mathcal{F} = \mathcal{G} \subset \textsc{GenEq}$.

Theorem

Every connected $\mathcal{F}$-gadget has a symmetric signature $\implies$ $\mathcal{F}$ is odeco

Lemma (proved)

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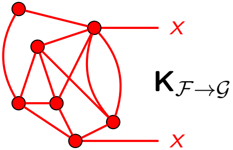

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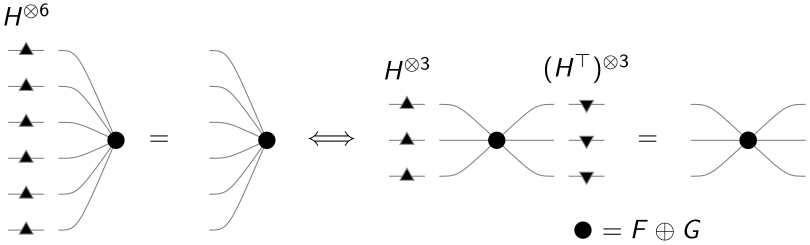
Proof of the Converse: A Diagonal Intertwiner

- Let F, G have arity $2n$.



$$H \in \mathrm{Stab}(F \oplus G) \iff H^{\otimes n}(F \oplus G)^{n,n} = (F \oplus G)^{n,n} H^{\otimes n} \iff$$

$$\begin{bmatrix} * & \cdots & * \\ \vdots & \iddots & \vdots \\ D^{\otimes n} & \cdots & * \end{bmatrix} \begin{bmatrix} F^{n,n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G^{n,n} \end{bmatrix} = \begin{bmatrix} F^{n,n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G^{n,n} \end{bmatrix} \begin{bmatrix} * & \cdots & * \\ \vdots & \iddots & \vdots \\ D^{\otimes n} & \cdots & * \end{bmatrix}$$

$$\implies D^{\otimes n} F^{n,n} = G^{n,n} D^{\otimes n}.$$