Planar #CSP Indistinguishability Corresponds to Quantum Isomorphism - A Holant Viewpoint

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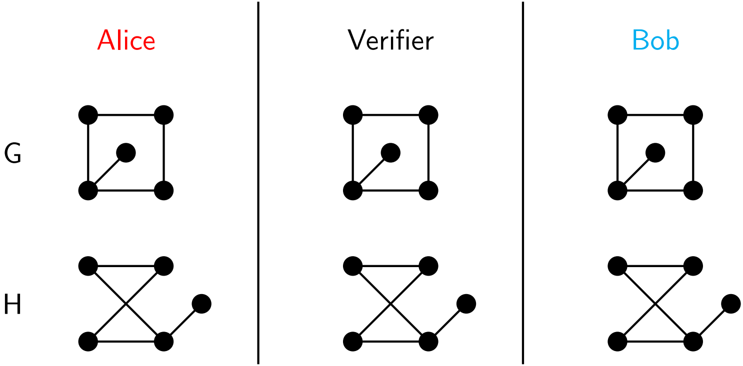

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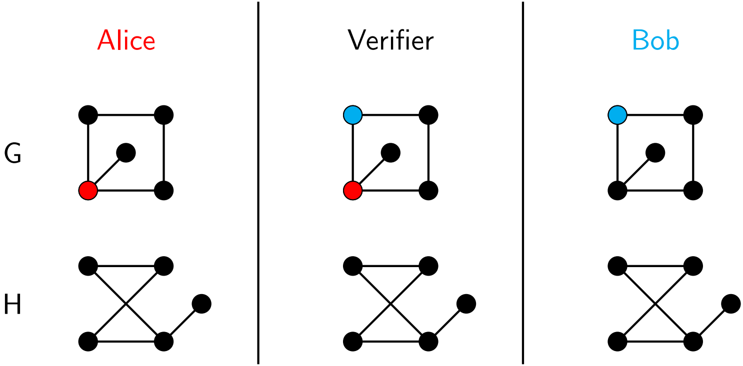

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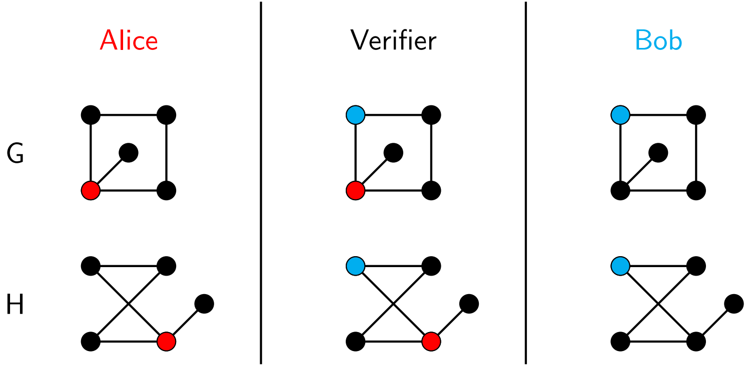

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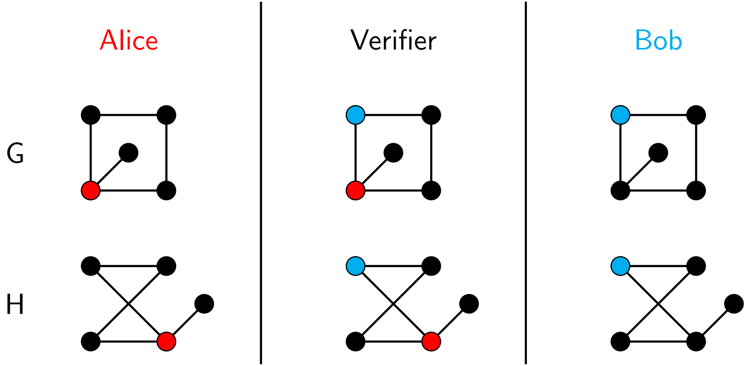

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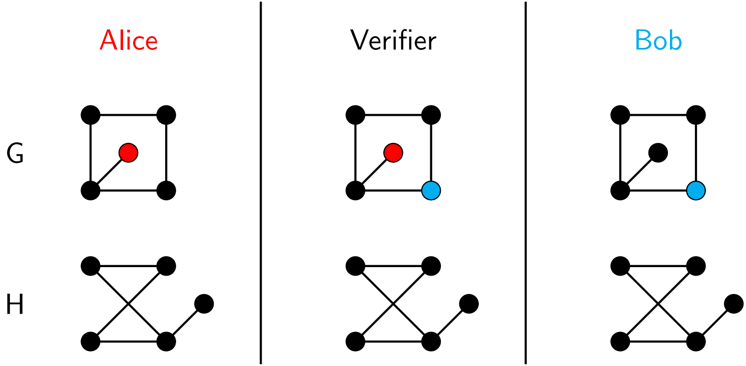

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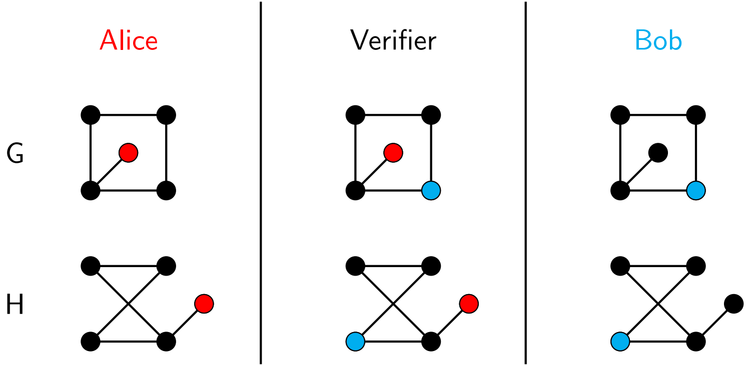

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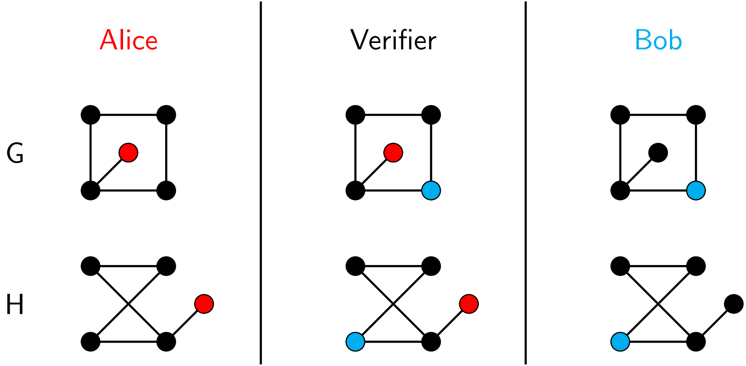

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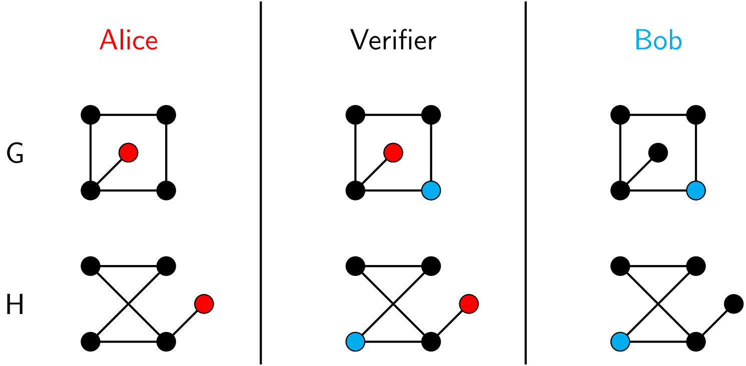

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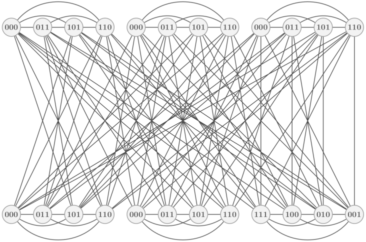

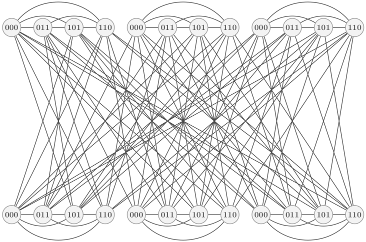

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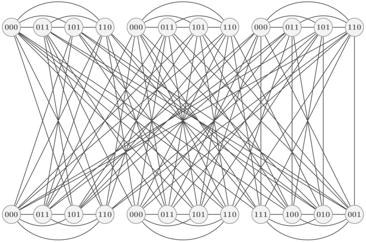

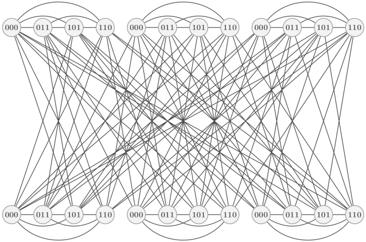

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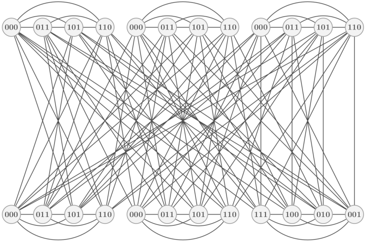
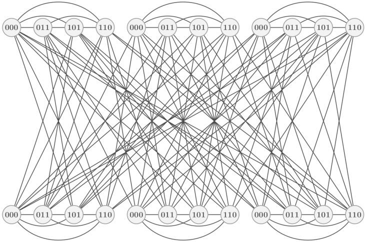
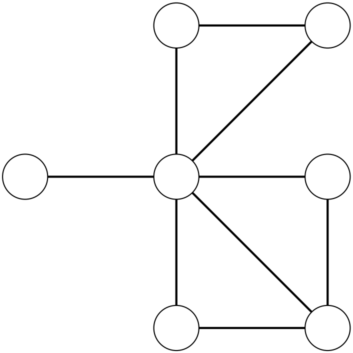




- Non-isomorphic Hadamard graphs [Chan and Martin '24]
- Quantum isomorphism is **undecidable** [AMRSSV'19, Slofstra'19]

Graph Homomorphism

A mapping $\phi : V(K) \to V(G)$ is a **graph homomorphism** if it maps all edges to edges: $\{u, v\} \in E(K) \implies \{\phi(u), \phi(v)\} \in E(G)$.



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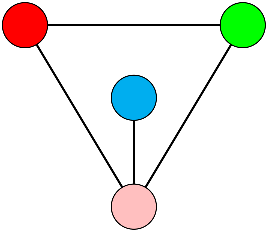

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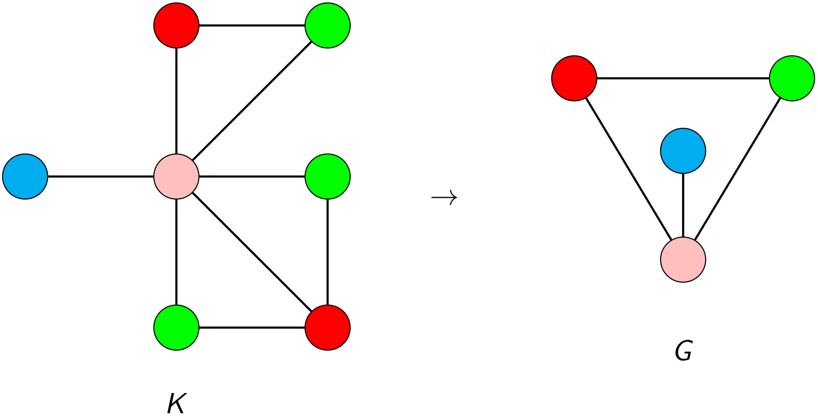

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- Goal: extend this result to #CSP

#CSP: the Counting Constraint Satisfaction Problem

Counting graph homomorphisms is a #CSP.

- **Constraint function** (tensor) $F : [q]^n \to \mathbb{R}$
  - **Domain** $[q] := \{0, 1, \ldots, q-1\}$
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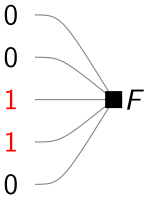

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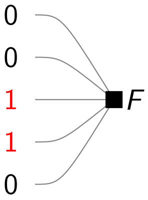

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- Let $\mathcal{F}$ be a set of constraint functions (all on same domain).

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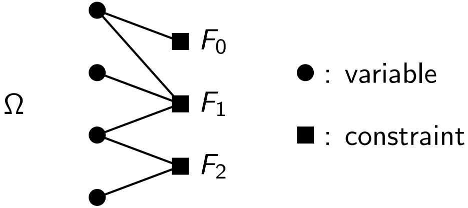

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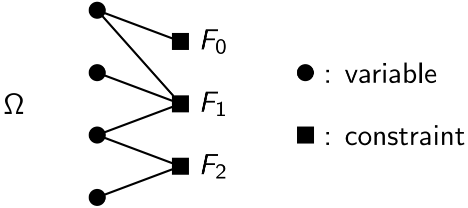

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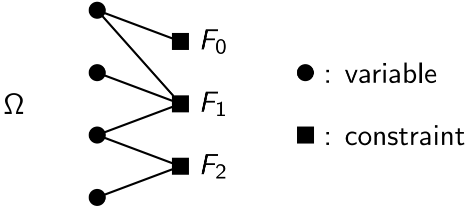

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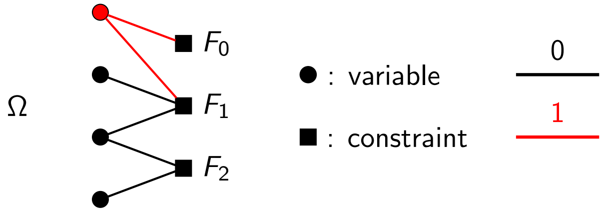

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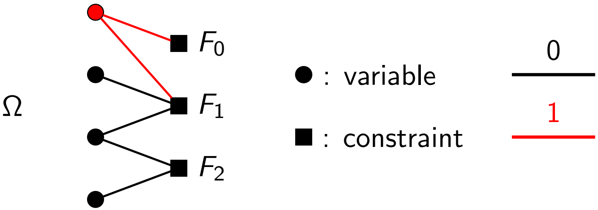

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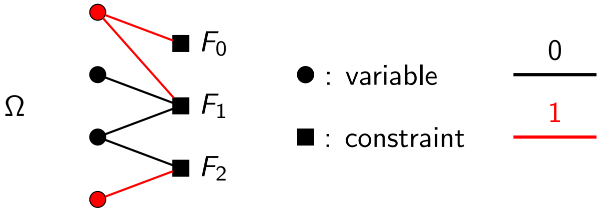

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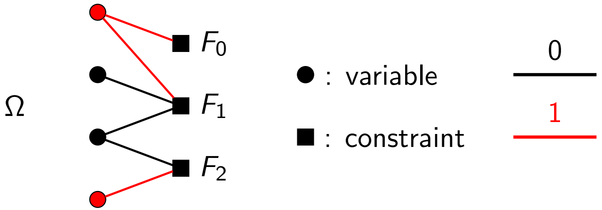

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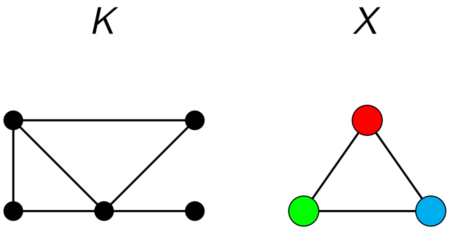

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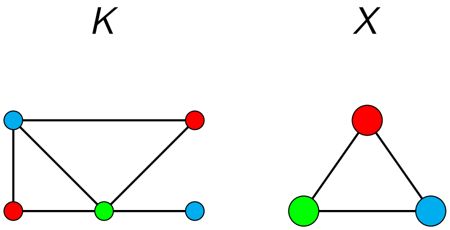

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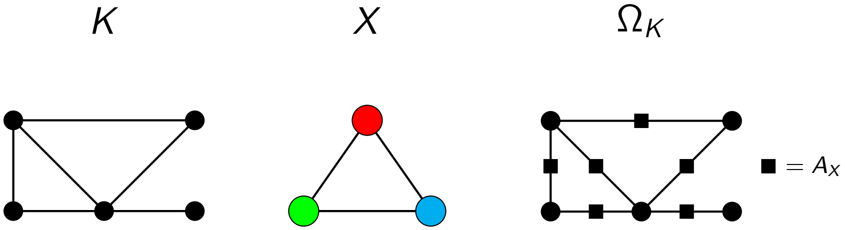

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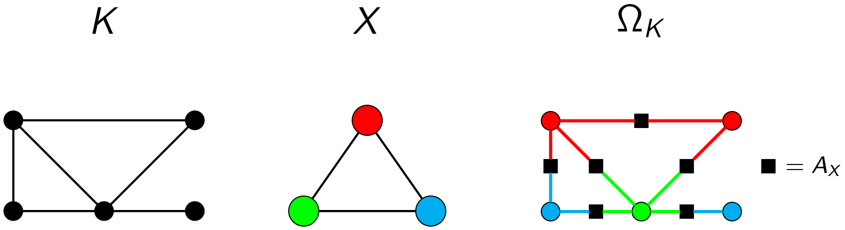

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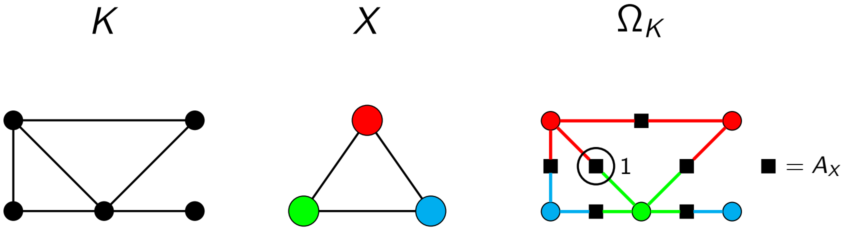

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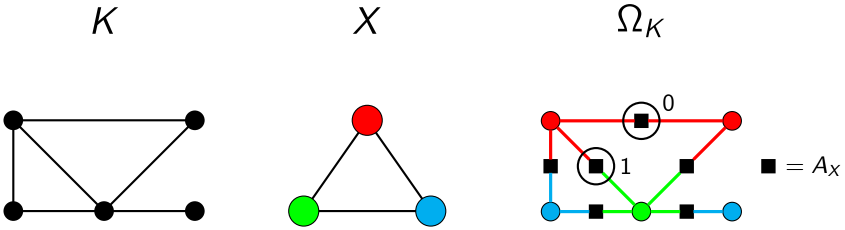

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Theorem (Cai-Fu'19)

For any set $\mathcal{F}$ of $\mathbb{C}$-valued constraint functions over Boolean variables, $\#$CSP$(\cdot, \mathcal{F})$ is exactly one of the following:

1. *P-time solvable;*
2. *P-time solvable over planar graphs but $\#$P-hard over general graphs;*
3. *$\#$P-hard over planar graphs.*

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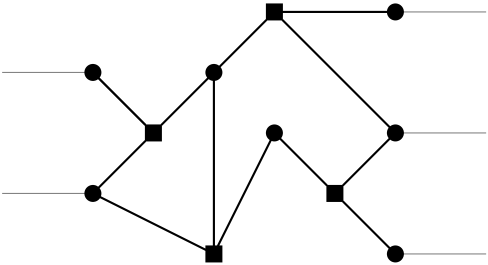

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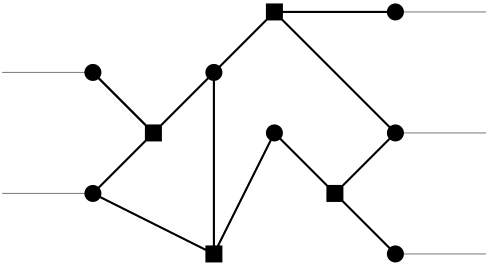

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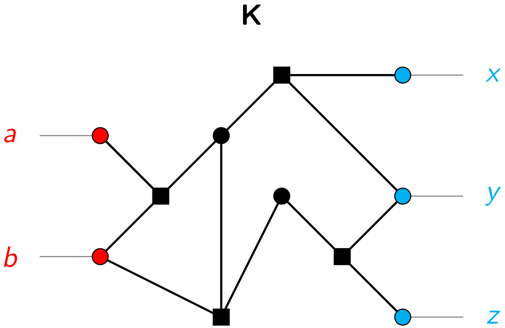

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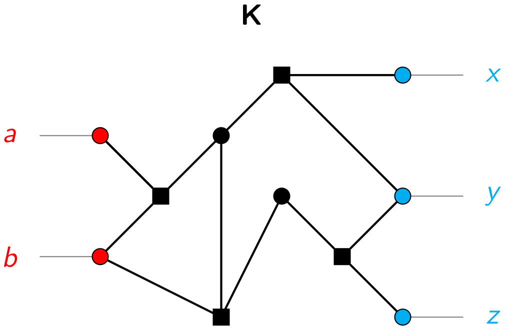

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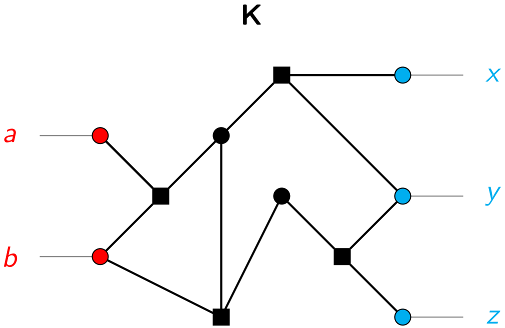

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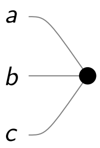
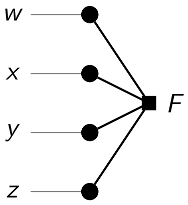
$\mathbf{E}^{3,0}$



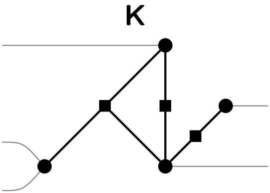
$= \delta_{abc}$

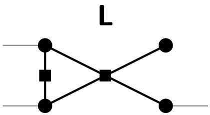
F

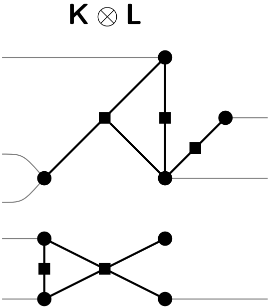


$= F(w, x, y, z)$

Gadget Operations







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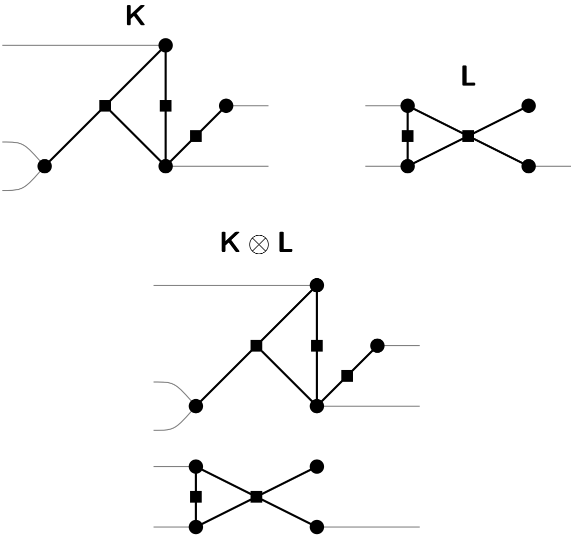

- $M(\mathbf{K} \otimes \mathbf{L}) = M(\mathbf{K}) \otimes M(\mathbf{L})$

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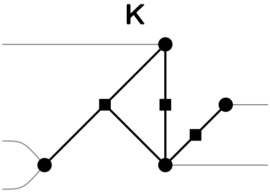

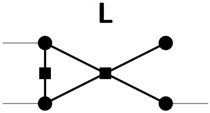

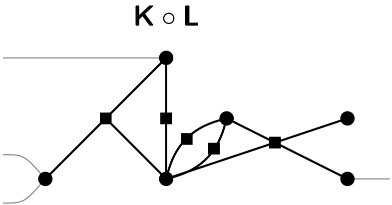

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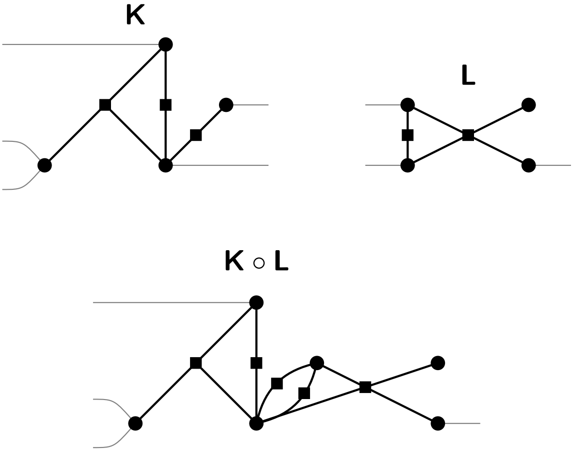

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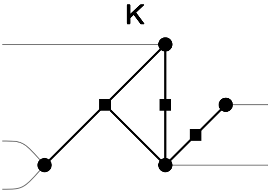

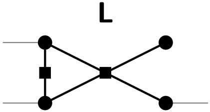

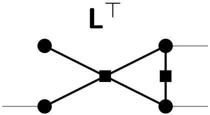

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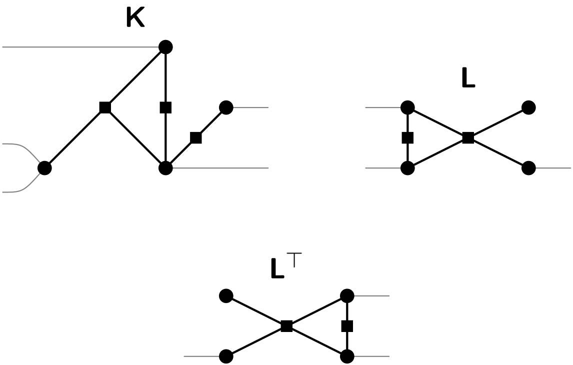

- $M(\mathbf{L}^\top) = M(\mathbf{L})^\top$

Quantum Permutation Matrices

Theorem (AMRSSV'18, Lupini-Mančinska-Roberson'17)

$G \cong_q H$ iff there is a **quantum permutation matrix** *U such that $UA_G U^\top = A_H$.*

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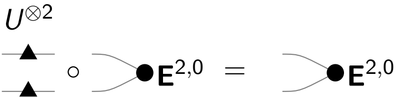
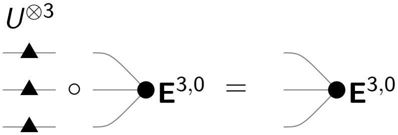


U is orthogonal: $UU^\top = I$



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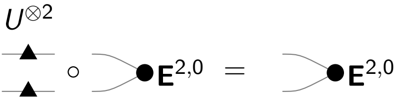

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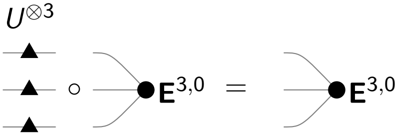

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- If the entries of U commute and $UA_G U^\top = A_H$, then $G \cong H$.

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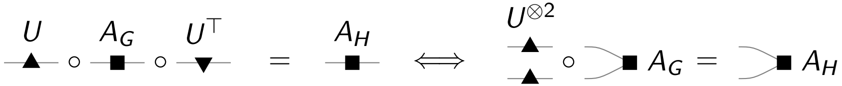

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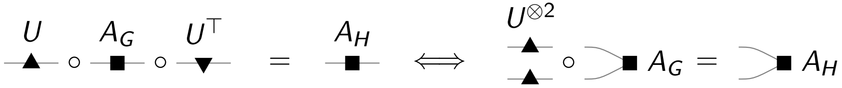

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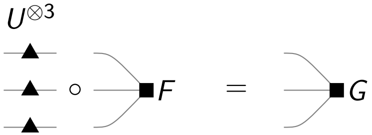

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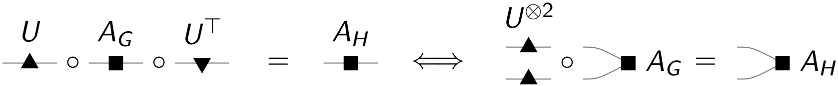

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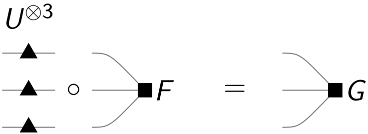

- $\mathcal{F} \cong_q \mathcal{G}$ if there is a U such that $U\mathcal{F} = \mathcal{G}$ (simultaneous action).

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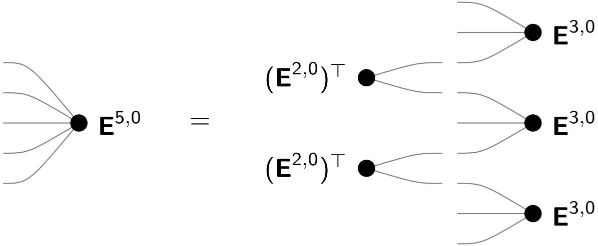
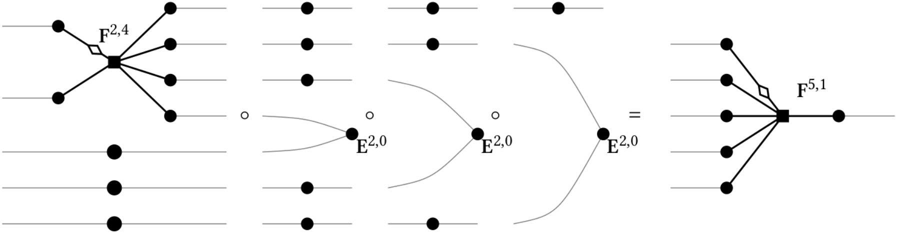
Constructing $\mathbf{E}^{n,0}$

- $\{\mathbf{E}^{n,0} \mid n \geq 1\} \subset \langle \mathbf{E}^{2,0}, \mathbf{E}^{3,0} \rangle_{\circ,\otimes,\top}$:



Edge Pivoting

- Every $\mathbf{K}^{\ell,r} \in \langle \mathbf{E}^{2,0}, \mathbf{E}^{1,1}, \mathbf{K}^{\ell',r'} \rangle_{\circ,\otimes,\top}$:



Nonplanar Gadget Decomposition

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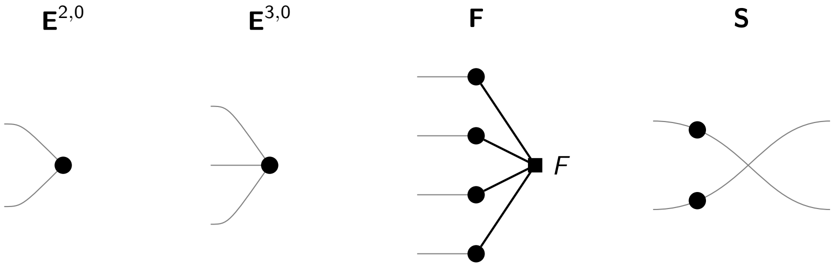
$\langle \mathbf{E}^{2,0}, \mathbf{E}^{3,0}, \{\mathbf{F} : F \in \mathcal{F}\}, \mathbf{S} \rangle_{\circ,\otimes,\top} =$ all $\#\mathrm{CSP}(\cdot, \mathcal{F})$ gadgets.



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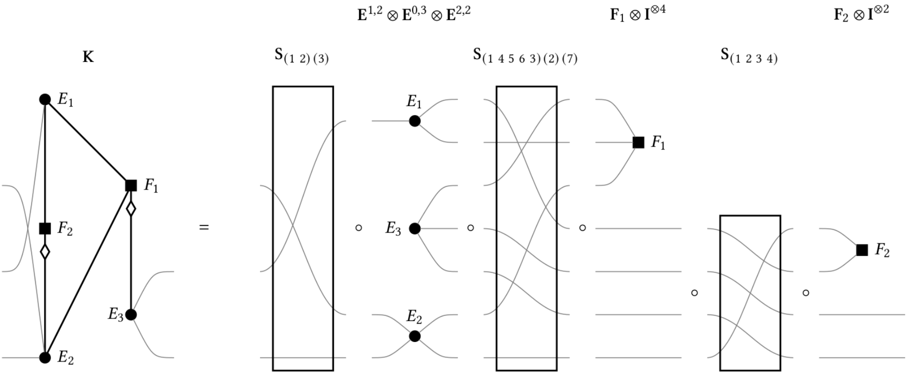

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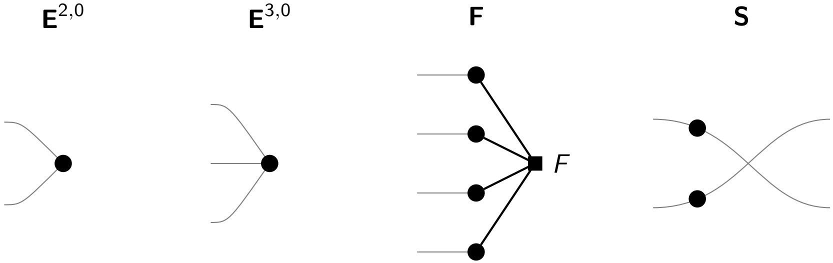

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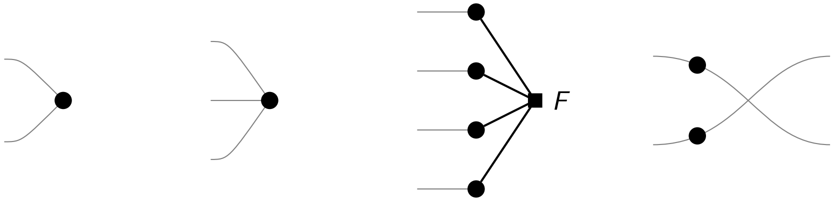
$\mathbf{E}^{2,0}$ $\mathbf{E}^{3,0}$ $\mathbf{F}$ $\mathbf{S}$



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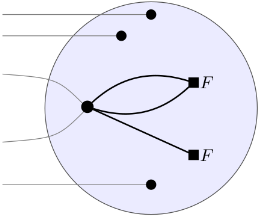

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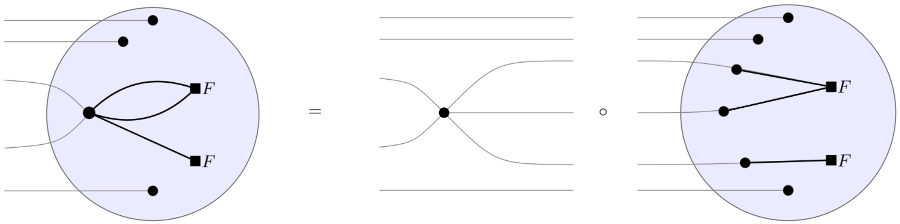

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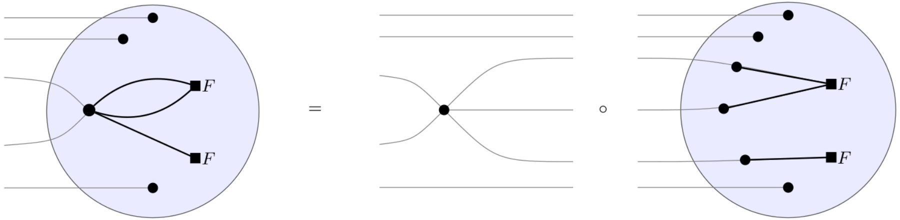

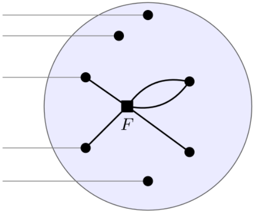

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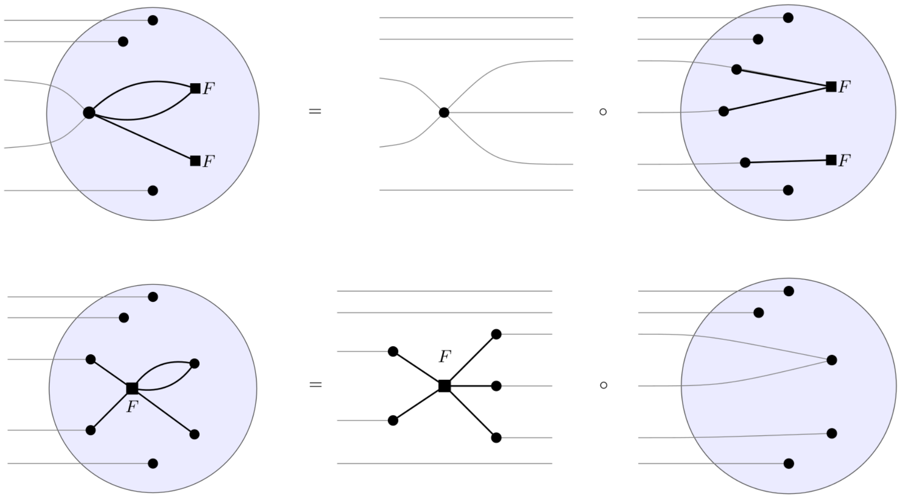

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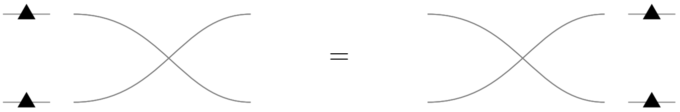
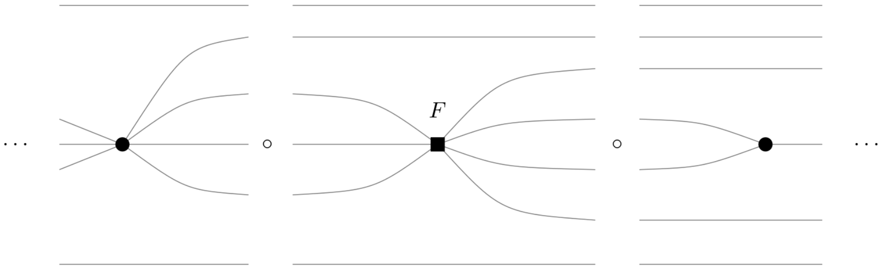
- $U^{\otimes 2}\, \mathbf{S} = \mathbf{S}\, U^{\otimes 2} \iff$ entries of U commute!



A Quantum Holographic Transformation



Decompose Ω into a composition of building block gadgets.

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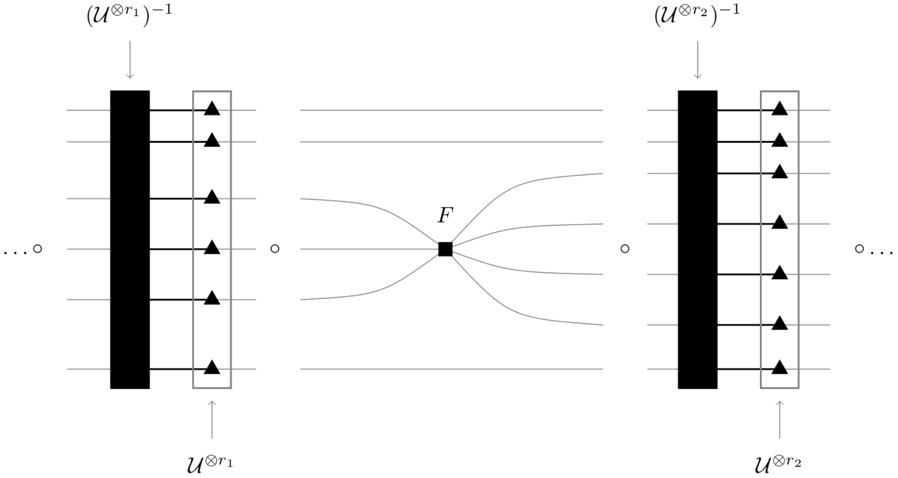

Insert $(U^{\otimes r_i})^{-1} U^{\otimes r_i} = I$ between the ith and $(i+1)$st factors.

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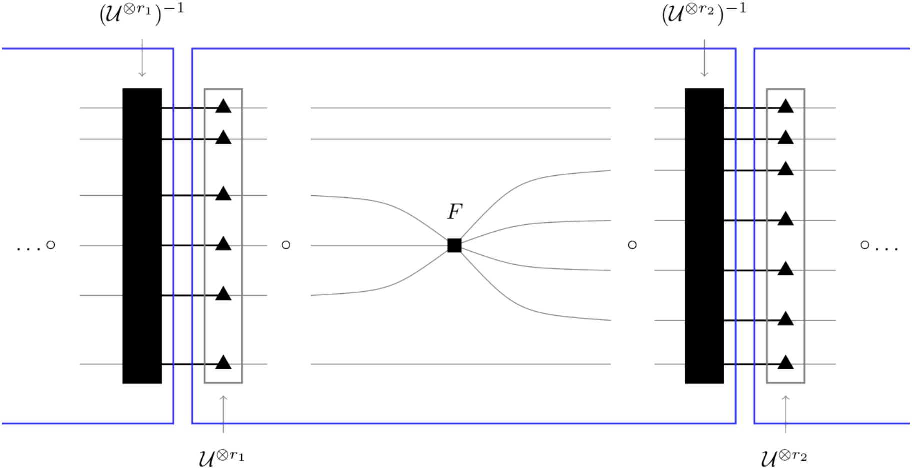

Reassociate. Now, U transforms F to G and preserves $\bullet$ **E**, so...

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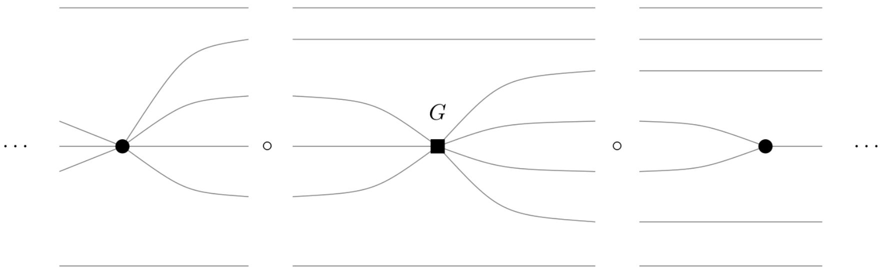

Every F is converted to G without changing the #CSP value:
$\#\text{CSP}(\Omega, \mathcal{F}) = \#\text{CSP}(\Omega, \mathcal{G})$.

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$$C_{\mathrm{Qut}(\mathcal{F})} = \left\{ \text{matrix } A \;\middle|\; U^{\otimes m} A = A U^{\otimes d} \right\}$$

Characterization of The Intertwiners

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- $F \cong_q G$.

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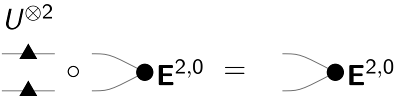

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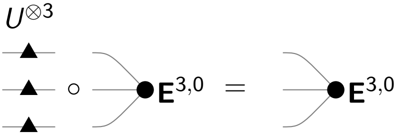

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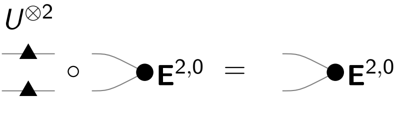

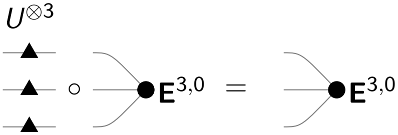

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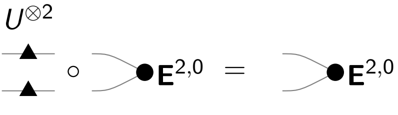

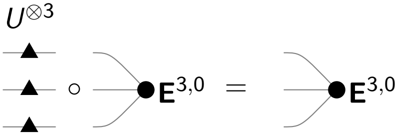

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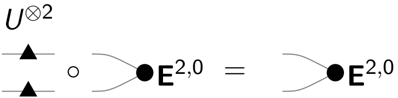

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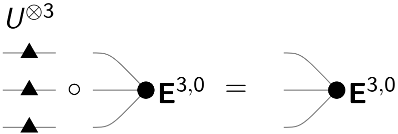

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- Why a generalization?

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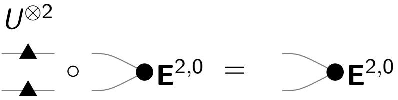

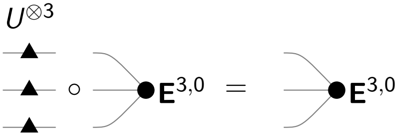

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- Add $M(\mathbf{E}^{3,0})$ to $\mathcal{F}$ and $\mathcal{G}$ as a constraint function.

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Corollary [Cai-Maran-Y.'26]

Undecidable: given G, x, y, is there a binary planar G-gadget $\mathbf{K}$ such that $M(\mathbf{K})_{xx} \neq M(\mathbf{K})_{yy}$?

Thank you!
Questions?