Counting Indistinguishability and the Converse of the Holant Theorem

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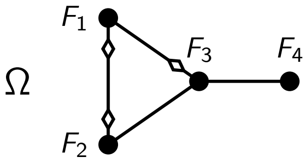
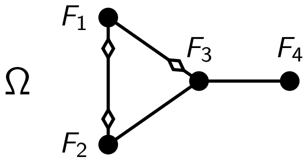
- **Signature** $F : [q]^n \to \mathbb{C}$
  - **Domain** $[q] := \{0, 1, \ldots, q-1\}$
  - **Arity** $n \geq 0$
- e.g. $q = 2$, $n = 3$: $F(x_1, x_2, x_3)$ for Boolean variables x_1, x_2, x_3.
- Let $\mathcal{F}$ be a set of signatures (all on same domain).
- $\mathcal{F}$-**grid** Ω is a multigraph with a signature from $\mathcal{F}$ on each vertex.
  - Arity of signature equals degree of vertex.
  - Order incident edges counterclockwise.





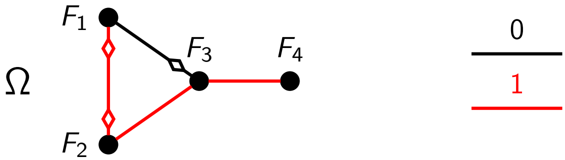
- Let F_v be the signature on vertex v.
- Goal: compute the **Holant value** of Ω:

$$\text{Holant}_{\mathcal{F}}(\Omega) = \sum_{\sigma : E(\Omega) \to [q]} \prod_{v \in V(\Omega)} F_v(\sigma(\text{edges incident to } v)).$$

- Example: domain $q = 2$:

$$\text{Holant}_{\mathcal{F}}(\Omega) = F_1(1, 0) \cdot F_2(1, 1) \cdot F_3(0, 1, 1) \cdot F_4(1) + \cdots$$

Holant Problems



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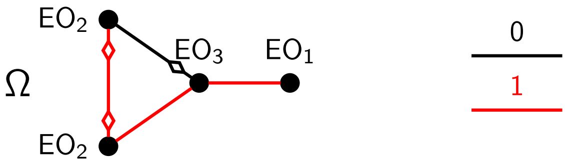
Example: Counting Perfect Matchings



- $EO_n : \{0,1\}^n \to \{0,1\}$ – **ExactOne** signature.
- $EO_n(x_1, \ldots, x_n) = 1$ iff exactly one $x_i = 1$.

$$\begin{aligned} \text{Holant}_{EO}(\Omega) = \; & EO_2(1,0) \cdot EO_2(1,1) \cdot EO_3(0,1,1) \cdot EO_1(1) + \\ & \ldots \\ = \; & 1 \cdot 0 \cdot 0 \cdot 1 + \\ & \ldots \end{aligned}$$

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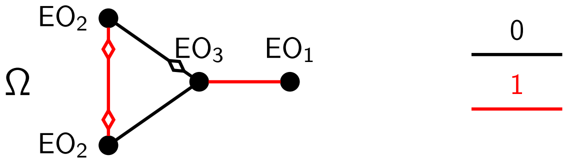

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& \ldots \\
=\;& 1 \cdot 0 \cdot 0 \cdot 1 + \\
& 1 \cdot 1 \cdot 1 \cdot 1 + \\
& \ldots
\end{aligned}
$$

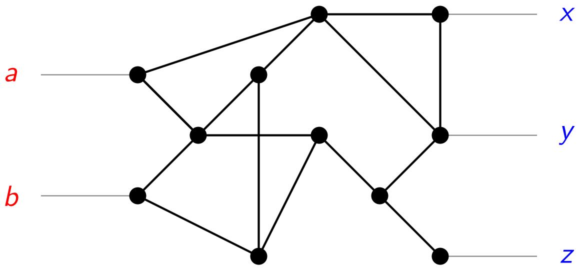
- Very expressive framework for counting problems.
- But restrictive enough to admit **complexity dichotomy theorems**:
- For any signature set $\mathcal{F}$, Holant$_{\mathcal{F}}$ is always either in P or #P-hard, with nothing in between.

Broad dichotomies exist for $\mathcal{F}$ containing signatures that are

- Domain $q = 2$, $\mathbb{C}$-valued, symmetric (Cai, Guo, and Williams [CGW16])
- Domain $q = 2$, $\mathbb{R}$-valued (Shao and Cai [SC20])
- Domain $q = 3$, $\mathbb{R}$-valued, symmetric Holant* (Cai and Ihm [CI25]).

Gadgets

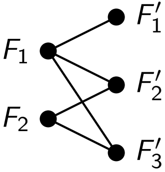
- A **gadget** is a signature grid with **dangling edges**.
- Here, signatures assembled into a 5-ary signature M.



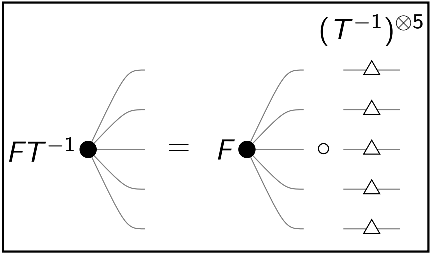
- $a, b, x, y, z \in [q]$.
- $M(a, b, x, y, z)$ is the Holant value with dangling edges fixed to a, b, x, y, z.

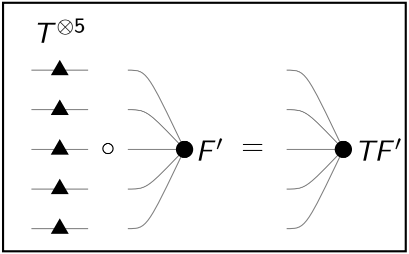
Bipartite Holant and Signature Transformations

- $\mathcal{F} \mid \mathcal{F}'$ denotes a **bipartite** Holant problem.
- $F_i \in \mathcal{F}$ is covariant (row vector)
- $F'_i \in \mathcal{F}'$ is contravariant (col vector)



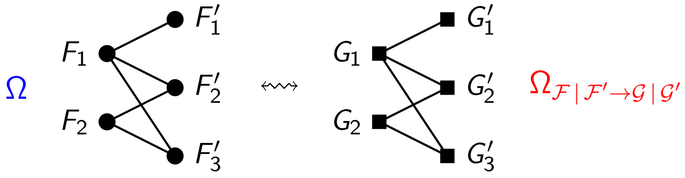
- Let $T \in \mathrm{GL}_q$ and F, F' on domain $[q]$.





- $\mathcal{F}\, T^{-1}$ and $T\, \mathcal{F}'$: simultaneous transformation by T.

- Let $\mathcal{F} \,|\, \mathcal{F}'$ and $\mathcal{G} \,|\, \mathcal{G}'$ be on the same domain $[q]$.
- Arity-respecting bijections $\mathcal{F} \leftrightsquigarrow \mathcal{G}$ and $\mathcal{F}' \leftrightsquigarrow \mathcal{G}'$.



Definition

$\mathcal{F} \,|\, \mathcal{F}'$ and $\mathcal{G} \,|\, \mathcal{G}'$ are **Holant-indistinguishable** if, for every $\mathcal{F} \,|\, \mathcal{F}'$-grid Ω,

$$\text{Holant}(\Omega) = \text{Holant}(\Omega_{\mathcal{F} \,|\, \mathcal{F}' \to \mathcal{G} \,|\, \mathcal{G}'}).$$

Theorem (The Holant Theorem [Val08])

If $\mathcal{F} \,|\, \mathcal{F}' = \mathcal{G}\, T^{-1} | T\, \mathcal{G}'$, then $\mathcal{F} \,|\, \mathcal{F}'$ and $\mathcal{G} \,|\, \mathcal{G}'$ are Holant-indistinguishable.

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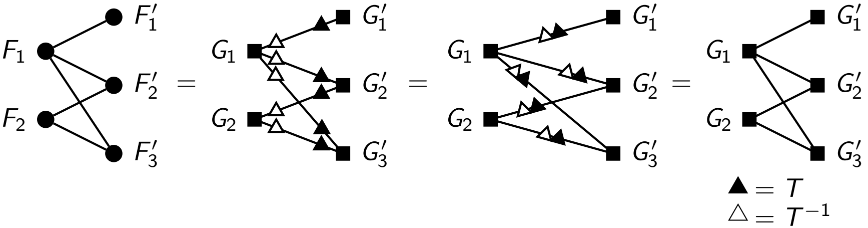


- Xia conjectured the converse of the Holant theorem [Xia10].
- Converse does not hold in general [CGW16].
- Holds for $\mathbb{R}$-valued $\mathcal{F}, \mathcal{G}$ and $\mathcal{F}' = \mathcal{G}' = \{I\}$, get **orthogonal** T [You25].
- We prove two near-converses generalizing the orthogonal case.

The Approximate Converse: Orbit Closure Intersection

- Apply techniques from geometric invariant theory:

Definition

The GL_q-*orbit closure* $\overline{\mathrm{GL}_q(\mathcal{F} \mid \mathcal{F}')}$ is the Euclidean closure of

$$\mathrm{GL}_q(\mathcal{F} \mid \mathcal{F}') = \{(\mathcal{F}\, T^{-1} | T\, \mathcal{F}) : T \in \mathrm{GL}_q\}.$$

Theorem (Mumford, Fogarty, Kirwan [MFK94])

$\mathcal{F} \mid \mathcal{F}'$ and $\mathcal{G} \mid \mathcal{G}'$ are indistinguishable under all GL_q-invariant polynomials iff $\overline{\mathrm{GL}_q(\mathcal{F} \mid \mathcal{F}')}$ and $\overline{\mathrm{GL}_q(\mathcal{G} \mid \mathcal{G}')}$ intersect.

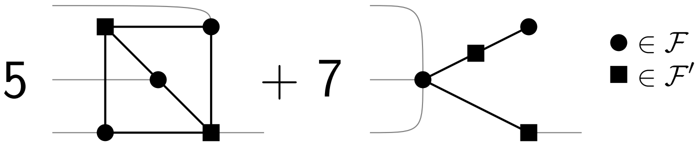
- Holant_Ω capture all GL_q-invariant polynomials!

Theorem (The Approximate Converse)

$\mathcal{F} \mid \mathcal{F}'$ and $\mathcal{G} \mid \mathcal{G}'$ are Holant-indistinguishable iff $\overline{\mathrm{GL}_q(\mathcal{F} \mid \mathcal{F}')}$ and $\overline{\mathrm{GL}_q(\mathcal{G} \mid \mathcal{G}')}$ intersect.

The Conditional Converse: Quantum Nonvanishing

- A **quantum gadget** is a formal linear combination of gadgets.



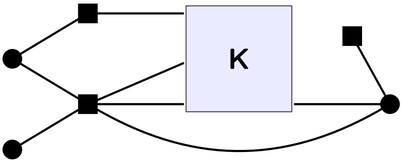
Definition

$\mathcal{F} \mid \mathcal{F}'$ is **quantum-nonvanishing** if $\forall$ quantum $\mathcal{F} \mid \mathcal{F}'$-gadget $\mathbf{K} \neq 0$, $\exists\ \mathcal{F} \mid \mathcal{F}'$-grid Ω containing $\mathbf{K}$ s.t. $\text{Holant}(\Omega) \neq 0$.

Example

$\mathcal{F} \mid \mathcal{F}' = [1\ i] \mid \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ is quantum-vanishing because every $\mathcal{F} \mid \mathcal{F}'$-grid has value 0:

$$[1\ i] \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right] = 1 \cdot 1 + i \cdot i = 0$$



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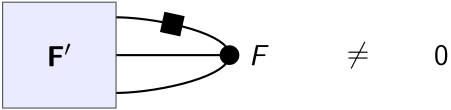
The Conditional Converse: Quantum Nonvanishing

Theorem (The Conditional Converse)

If $\mathcal{F} \mid \mathcal{F}'$ and $\mathcal{G} \mid \mathcal{G}'$ are Holant-indistinguishable & quantum-nonvanishing, then there is a $T \in \mathrm{GL}_q$ such that $\mathcal{F} \mid \mathcal{F}' = \mathcal{G}\, T^{-1} \mid T\, \mathcal{G}'$.

- Proof uses an invariant-theoretic characterization of quantum gadget signatures (Derksen and Makam [DM23]).



- By Derksen and Makam's theorem, can find a gadget with $\lambda_1 \neq \lambda_2$.
- Interpolate **subdomain restrictors** $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.
- Repeat on the subdomains.

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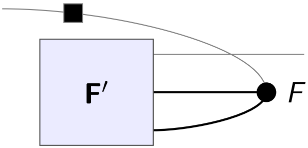

$$= \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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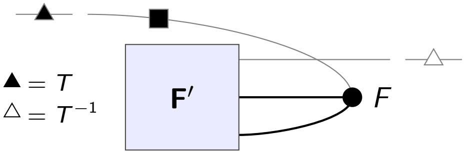

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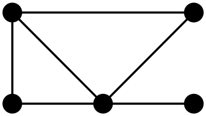
Counting Graph Homomorphisms with Bipartite Holant

- $\phi : V(K) \to V(X)$ is a **graph homomorphism** if it maps every edge of K to an edge of X.
- $\mathcal{EQ} = \{=_n \mid n \in \mathbb{N}\}$ where

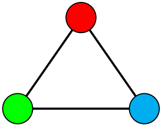
$$(=_n)(x_1, \ldots, x_n) = \begin{cases} 1 & x_1 = \ldots = x_n \\ 0 & \text{otherwise} \end{cases}$$

- $\#\text{hom}(K, X) = \text{Holant}_{\{A_X\}|\mathcal{EQ}}(\Omega)$:

K



X



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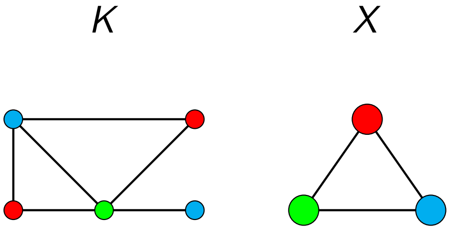

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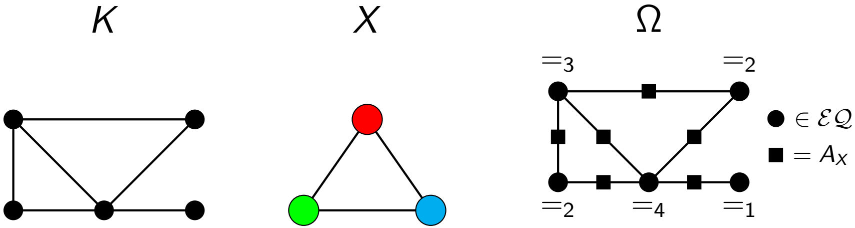

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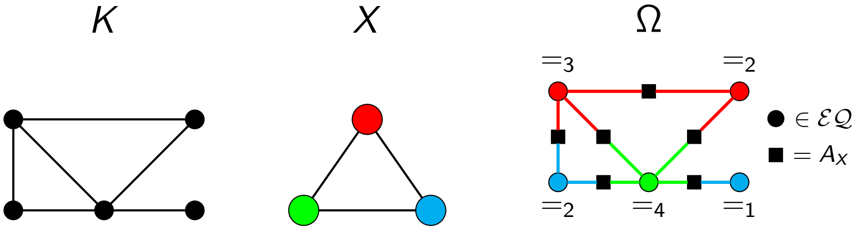

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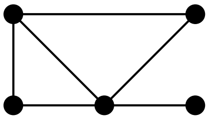
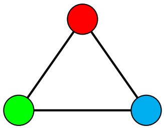
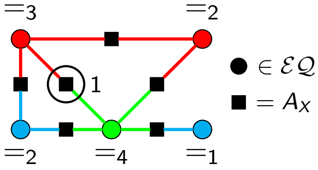


X



Ω



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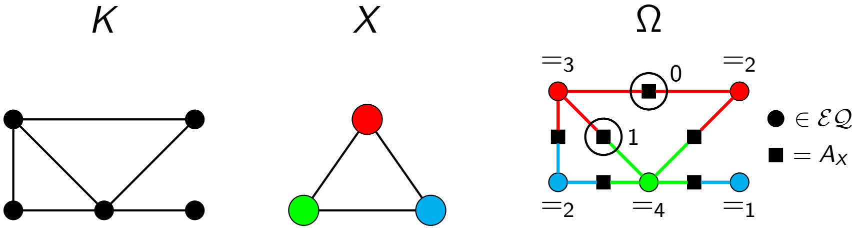


Homomorphism Indistinguishability

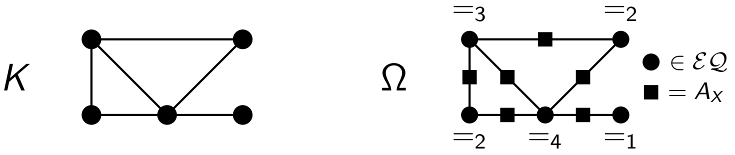
- Graphs X and Y are **homomorphism-indistinguishable** over $\mathfrak{G}$ if $\#\mathrm{hom}(K, X) = \#\mathrm{hom}(K, Y)$ for every $K \in \mathfrak{G}$.

Theorem (Lovász [Lov67])

X and Y are isomorphic iff X and Y are homomorphism-indistinguishable over all graphs.

- Get relaxations of isomorphism for other $\mathfrak{G}$
  - e.g. trees, cycles, planar graphs, bounded tree/pathwidth, etc. [Sep24]
- Roberson [Rob22] showed that graphs of degree $\leq d$ does not induce isomorphism for any d,
- but the exact indistinguishability relation remained open.

- Recall: $\#\mathrm{hom}(\cdot, X) \longleftrightarrow \mathrm{Holant}_{\{A_X\}|\mathcal{EQ}}$.



- $\mathcal{EQ}_{\leq d} \subset \mathcal{EQ}$ is equalities of arity $\leq d$.
- $\mathrm{Holant}_{\{A_X\}|\mathcal{EQ}_{\leq d}}$ counts homomorphisms from graphs of degree $\leq d$.

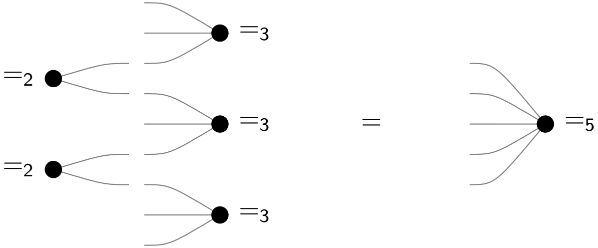
Theorem (The Approximate Converse)

$\mathcal{F} \mid \mathcal{F}'$ and $\mathcal{G} \mid \mathcal{G}'$ are Holant-indistinguishable iff $\overline{\mathrm{GL}_q(\mathcal{F} \mid \mathcal{F}')}$ and $\overline{\mathrm{GL}_q(\mathcal{G} \mid \mathcal{G}')}$ intersect.

Corollary

X and Y are homomorphism-indistinguishable over graphs of degree $\leq d$ iff $\overline{\mathrm{GL}_q(\{A_X\}|\mathcal{EQ}_{\leq d})}$ and $\overline{\mathrm{GL}_q(\{A_Y\}|\mathcal{EQ}_{\leq d})}$ intersect. This is decidable.

- $T\,\mathcal{EQ} = \mathcal{EQ} \iff T$ is a permutation matrix.
- Every $=_n$ is gadget-constructible from $=_2$ and $=_3$:



- $T\,\mathcal{EQ}_{\leq 3} = \mathcal{EQ}_{\leq 3} \iff T$ is a permutation matrix.

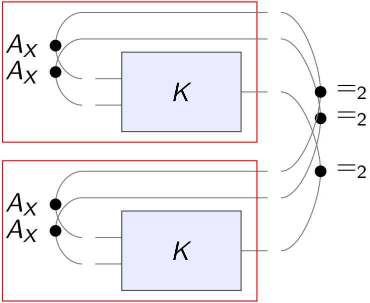
Homomorphisms from Graphs of Bounded Degree

- $T\,\mathcal{EQ}_{\leq 3} = \mathcal{EQ}_{\leq 3} \iff T$ is a permutation matrix.
- $\text{Holant}_{\{A_X\}|\mathcal{EQ}_{\leq 3}}$ counts homomorphisms from graphs of degree ≤ 3.

Theorem (The Conditional Converse)

If $\mathcal{F} \mid \mathcal{F}'$ and $\mathcal{G} \mid \mathcal{G}'$ are Holant-indistinguishable & quantum-nonvanishing, then there is a $T \in \mathrm{GL}_q$ such that $\mathcal{F} \mid \mathcal{F}' = \mathcal{G}\, T^{-1} | T\, \mathcal{G}'$.

- If A_X is invertible, then $\{A_X\} \mid \mathcal{EQ}_{\leq 3}$ is quantum-nonvanishing:
- Apply Theorem to $\{A_X\}|\mathcal{EQ}_{\leq 3}$ and $\{A_Y\}|\mathcal{EQ}_{\leq 3}$.



Corollary

If A_X, A_Y are invertible, then X and Y are homomorphism-indistinguishable over graphs of degree ≤ 3 iff $X \cong Y$.

Thank you!
Questions?

Jin-Yi Cai, Heng Guo, and Tyson Williams.
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SIAM Journal on Computing, 45(5):1671–1728, 2016.

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