The Converse of the Real Orthogonal Holant Theorem

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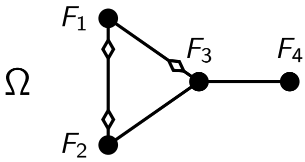
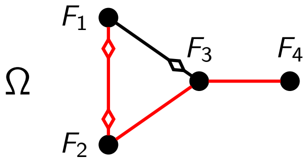
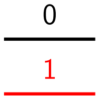
Signature Grids

- **Signature** $F : [q]^n \to \mathbb{R}$
  - **Domain** $[q] := \{0, 1, \ldots, q-1\}$
  - **Arity** $n \geq 0$
- e.g. $q = 2$, $n = 3$: $F(x_1, x_2, x_3)$ for Boolean variables x_1, x_2, x_3.
- Let $\mathcal{F}$ be a set of signatures (all on same domain).
- $\mathcal{F}$-**grid** Ω is a multigraph with a signature from $\mathcal{F}$ on each vertex
  - Arity of signature equals degree of vertex



Holant Problems





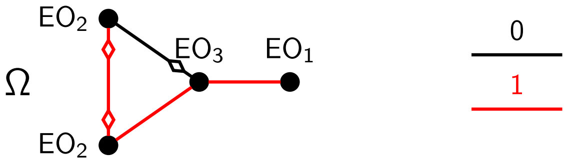
- Let F_v be the signature on vertex v.
- Order edges incident to v counterclockwise.
- Goal: compute the **Holant value** of Ω:

$$\text{Holant}_{\mathcal{F}}(\Omega) = \sum_{\sigma : E(\Omega) \to [q]} \prod_{v \in V(\Omega)} F_v(\sigma(\text{edges incident to } v)).$$

- Example: domain $q = 2$:

$$\text{Holant}_{\mathcal{F}}(\Omega) = F_1(1, 0) \cdot F_2(1, 1) \cdot F_3(0, 1, 1) \cdot F_4(1) + \dots$$

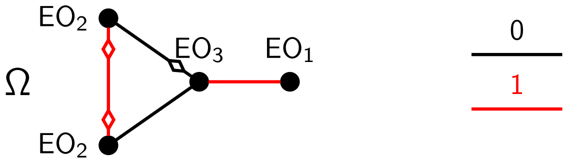
Example: Counting Perfect Matchings



- $EO_n : \{0,1\}^n \to \{0,1\}$ – **ExactOne** signature.
- $EO_n(x_1, \ldots, x_n) = 1$ iff exactly one $x_i = 1$.

$$\begin{aligned} \text{Holant}_{\text{EO}}(\Omega) &= \text{EO}_2(1,0) \cdot \text{EO}_2(1,1) \cdot \text{EO}_3(0,1,1) \cdot \text{EO}_1(1) + \ldots \\ &= 1 \cdot 0 \cdot 0 \cdot 1 + \ldots \end{aligned}$$

Example: Counting Perfect Matchings



- $EO_n : \{0,1\}^n \to \{0,1\}$ – **ExactOne** signature.
- $EO_n(x_1, \ldots, x_n) = 1$ iff exactly one $x_i = 1$.

$$\begin{aligned}
\text{Holant}_{\text{EO}}(\Omega) = \;& EO_2(1,0) \cdot EO_2(1,1) \cdot EO_3(0,1,1) \cdot EO_1(1) + \\
& EO_2(1,0) \cdot EO_2(1,\mathbf{0}) \cdot EO_3(0,\mathbf{0},1) \cdot EO_1(1) + \\
& \ldots \\
= \;& 1 \cdot 0 \cdot 0 \cdot 1 + \\
& 1 \cdot 1 \cdot 1 \cdot 1 \; + \\
& \ldots
\end{aligned}$$

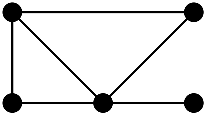
Example: Counting Graph Homomorphisms

- $\phi : V(K) \to V(X)$ is a **graph homomorphism** if it maps every edge of K to an edge of X.
- $\mathcal{EQ} = \{=_n \mid n \in \mathbb{N}\}$ where

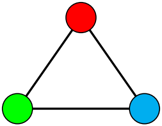
$$(=_n)(x_1, \ldots, x_n) = \begin{cases} 1 & x_1 = \ldots = x_n \\ 0 & \text{otherwise} \end{cases}$$

- $(\#\text{ homomorphisms } K \to X) = \text{Holant}_{\{A_X\} \cup \mathcal{EQ}}(\Omega)$:
  - Domain $[q] = V(X)$. Here $q = 3$.

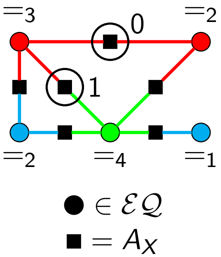
K



X



Ω



Why study Holant?

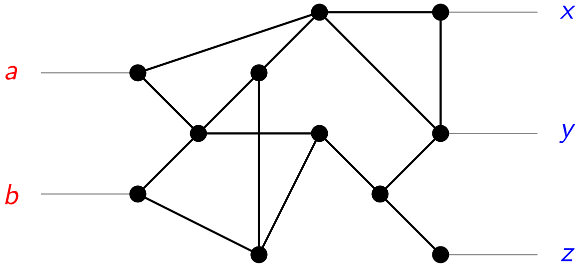
- Very expressive framework for counting problems.
- But restrictive enough to admit **complexity dichotomy theorems**:
- For any signature set $\mathcal{F}$ (of a certain class), $\text{Holant}_{\mathcal{F}}$ is always either in P or #P-hard, with nothing in between.

Broad dichotomies exist for $\mathcal{F}$ containing signatures that are

- Domain $q = 2$, $\mathbb{C}$-valued, symmetric (Cai, Guo, and Williams [CGW16])
- Domain $q = 2$, $\mathbb{R}$-valued (Shao and Cai [SC20])
- Domain $q = 3$, $\mathbb{R}$-valued, symmetric Holant^* (Cai and Ihm [CI25]).

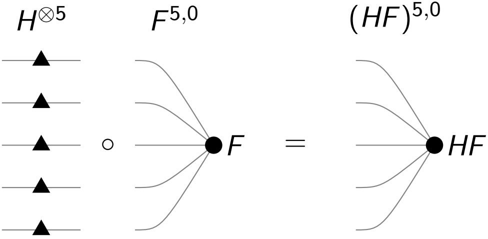
Gadgets

- A **gadget** is a signature grid with **dangling edges**.
- Here, signatures assembled into a 5-ary signature M.



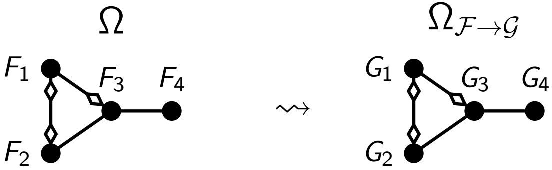
- $a, b, x, y, z \in [q]$.
- $M(a, b, x, y, z)$ is the Holant value with dangling edges fixed to a, b, x, y, z.
  - Think of M as a $2^q \times 3^q$ matrix.

- Let H be invertible $q \times q$ matrix, $F : [q]^n \to \mathbb{R}$.
- Define $HF : [q]^n \to \mathbb{R}$ by applying H to each input of F:



- HF is F under basis H.
- For signature set $\mathcal{F}$, define $H\mathcal{F} := \{HF \mid F \in \mathcal{F}\}$.

- Let $\mathcal{F}$ and $\mathcal{G}$ be signature sets on same domain $[q]$.
- Assume there is a bijection between $\mathcal{F}$ and $\mathcal{G}$ preserving arity.



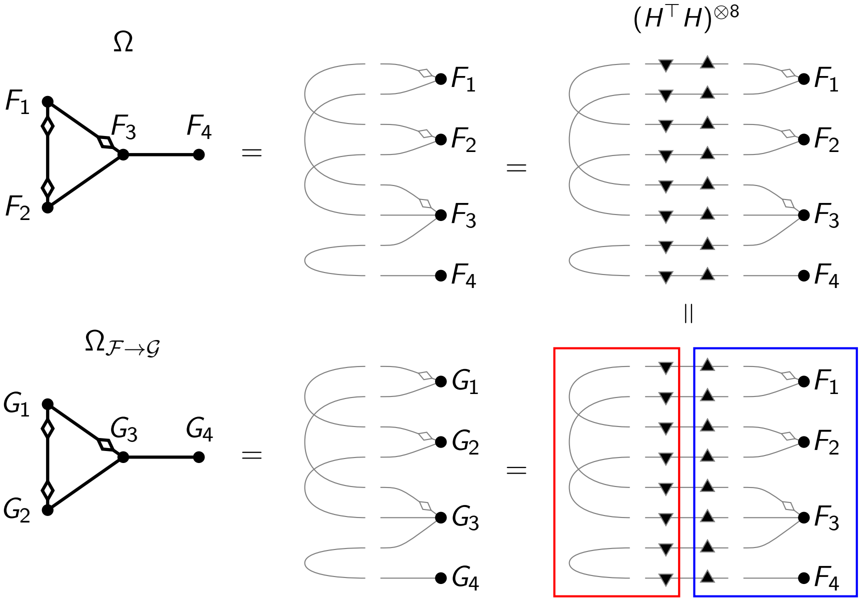
Definition

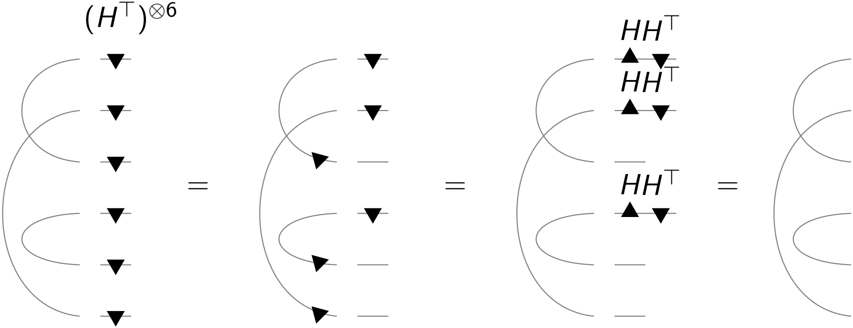
$\mathcal{F}$ and $\mathcal{G}$ are **Holant-indistinguishable** if, for every $\mathcal{F}$-grid Ω,

$$\text{Holant}(\Omega) = \text{Holant}(\Omega_{\mathcal{F}\to\mathcal{G}}).$$

Theorem (The Orthogonal Holant Theorem)

If $\mathcal{G} = H\mathcal{F}$ for orthogonal H, then $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.


Ω
$(H^\top H)^{\otimes 8}$
F_1 F_2 F_3 F_4
$\Omega_{\mathcal{F}\to\mathcal{G}}$
G_1 G_2 G_3 G_4


$(H^\top)^{\otimes 6}$
=
=
$HH^\top$
$HH^\top$
$HH^\top$
=

Main Result

Theorem (The Orthogonal Holant Theorem)

If $\mathcal{G} = H\mathcal{F}$ for orthogonal H, then $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

- Special case of Valiant's general Holant theorem. [Val08]
- *Holographic algorithms* using the Holant theorem are the original motivation for Holant problems.
- Xia conjectured the converse of the Holant theorem [Xia10].
- Converse does not hold in general [CGW16]
- But we show it does in the orthogonal case:

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for orthogonal H iff $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

Very Rough Proof Sketch

Theorem (Main Result)

$\mathcal{G} = H\mathcal{F}$ for orthogonal H iff $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.

- Novel technique: proof by induction on q (the domain size).

1. $\exists$ *some* matrix $H \neq 0$ (maybe not orthogonal) 'transforming' $\mathcal{F}$ to $\mathcal{G}$.
   - Uses invariant-theoretic result of Schrijver [Sch08].
2. Let $H = U^T DV$ be SVD of H.
3. Transform $\mathcal{F}$ by V and $\mathcal{G}$ by U to replace H with diagonal D.
4. $D = cI \implies D$ is orthogonal (up to scalar), and transforms $\mathcal{F}$ to $\mathcal{G}$.
5. If $D \neq cI$, then interpolate $\begin{bmatrix} I & \\ & 0 \end{bmatrix}$ from D.
6. Use $\begin{bmatrix} I & \\ & 0 \end{bmatrix}$ and $I - \begin{bmatrix} I & \\ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \\ & I \end{bmatrix}$ to break into two subdomains and apply induction.

Counting Indistinguishability

- This is a **counting indistinguishability theorem**.
- Many examples for counting homomorphisms to graphs X and Y.

Theorem (Lovász [Lov67])

X and Y are isomorphic iff X and Y are homomorphism-indistinguishable.

Theorem

$HA_X = A_Y H$ for some orthogonal H iff X and Y are homomorphism-indistinguishable over all cycles.

Theorem (Combination of above and [DGR18])

$HA_X = A_Y H$ for some orthogonal pseudo-stochastic H iff X and Y are homomorphism-indistinguishable over all cycles and paths.

- These all follow from our main result.
- Get a sharper version of Lovász' theorem:

Theorem (Lovász [Lov67])

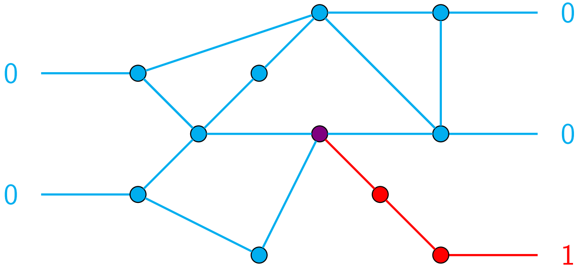
X and Y are isomorphic iff X and Y are homomorphism-indistinguishable.

Theorem

X and Y are isomorphic iff X and Y are homomorphism-indistinguishable over even-degree graphs.

- Recall: $\text{Holant}_{\{A_X\} \cup \mathcal{EQ}}$ counts homomorphisms to X.
- Similarly, $\text{Holant}_{\{A_Y\} \cup \mathcal{EQ}}$ counts homomorphisms to Y.
- By our theorem, $\exists H$ s.t. $H\{A_X\} = \{A_Y\}$ and $H\mathcal{EQ} = \mathcal{EQ}$.
- $H\mathcal{EQ} = \mathcal{EQ} \iff H$ is a permutation matrix.
- Let $\mathcal{EQ}_2 \subset \mathcal{EQ}$ be the signatures of even arity.
- $\text{Holant}_{\{A_X\} \cup \mathcal{EQ}_2}$: homomorphisms from even-degree graphs to X.
- $H\mathcal{EQ}_2 = \mathcal{EQ}_2 \iff H$ is a signed permutation matrix.
- Signed isomorphism $\iff$ isomorphism (A_X and A_Y are unsigned).

- $E \in \mathcal{GEQ}$: $E(x_1, \ldots, x_n) = 0$ unless $x_1 = \ldots = x_n$.
- Every connected $\mathcal{GEQ}$-gadget has a signature in $\mathcal{GEQ}$:



Odeco Signature Sets

- Every connected $\mathcal{GEQ}$-gadget has a signature in $\mathcal{GEQ}$
- Every connected $\mathcal{GEQ}$-gadget has a symmetric signature.
- We show $\mathcal{GEQ}$ is the **only** signature set with this $\uparrow$ property:
  - (up to orthogonal transformation)

Definition

$\mathcal{F}$ is **odeco** if $\exists$ orthogonal H such that $H\mathcal{F} \subset \mathcal{GEQ}$.

Theorem

$\mathcal{F}$ is odeco $\iff$ every connected $\mathcal{F}$-gadget has a symmetric signature.

- Extends characterization of [Rob16, BDHR17] from a single tensor to a set.

Thank you!
Questions?

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