Planar #CSP Equality Corresponds to Quantum Isomorphism - A Holant Viewpoint

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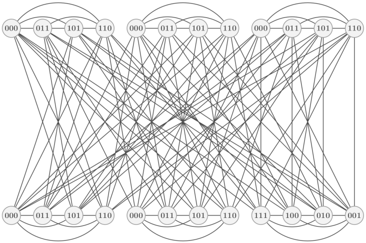
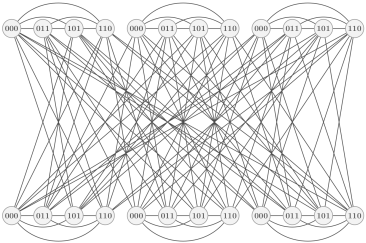
Graph isomorphism game

- Nonlocal game introduced in [AMRSSV'19]
- Players Alice and Bob cooperate to convince a referee that graphs X and Y are isomorphic.
- They are separated without communication.
- Alice gets a vertex of X or Y, responds with a vertex of other graph.
- Bob gets a vertex of X or Y, responds with a vertex of other graph.
- Players win if the two X vertices and the two Y vertices have the same relationship (equal, adjacent, or non-adjacent).
- $X \cong Y \iff$ the players have a perfect winning strategy.

Quantum graph isomorphism game

- Quantum strategy: Alice and Bob can measure a shared entangled quantum state after receiving their inputs.
  - Can choose quantum state ahead of time, but still can't communicate during game.
- X and Y are quantum isomorphic ($X \cong_{qc} Y$) if Alice and Bob have a perfect quantum winning strategy.
- There exist X, Y such that $X \cong_{qc} Y$ but $X \ncong Y$! [AMRSSV'19]

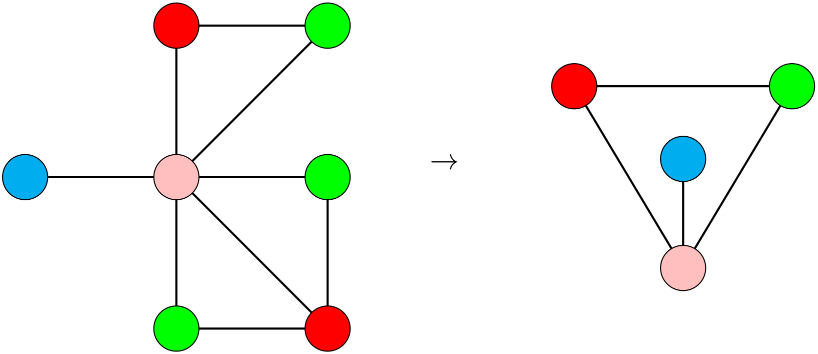




Graph Homomorphism

A mapping $\phi : V(X) \to V(Y)$ is a graph homomorphism if it preserves adjacency: $\{u, v\} \in E(X) \implies \{\phi(u), \phi(v)\} \in E(Y)$.

(i.e., Edges are mapped to edges)



Homomorphism Theorems

Let $\hom(G, X)$ be the number of graph homomorphisms from G to X.

Theorem (Lovász'67)

$X \cong Y$ *iff* $[\ \hom(G, X) = \hom(G, Y)\ \forall$ *graph* $G]$.

Theorem (Mančinska-Roberson'20)

$X \cong_{qc} Y$ *iff* $[\ \hom(G, X) = \hom(G, Y)\ \forall$ **planar** *graph* $G]$.

What is the connection between planarity and quantum nonlocal games?

Quantum permutation matrices

- $X \cong Y$ iff there is a permutation matrix P s.t. $PA_X P^{-1} = A_Y$.

A **quantum permutation matrix** is an abstract relaxation of a permutation matrix.

- Entries come from a noncommutative C^*-algebra, but
- Rows and columns still add up to 1
- Product of distinct elements in same row or column in 0

Theorem (Lupini-Mančinska-Roberson'17)

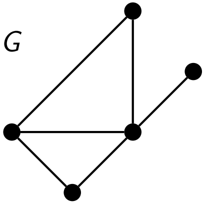
$X \cong_{qc} Y$ iff there is a quantum permutation matrix $\mathcal{U}$ s.t. $\mathcal{U} A_X \mathcal{U}^{-1} = A_Y$.

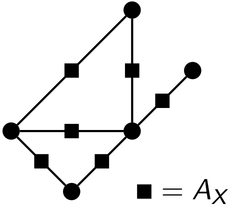
Higher dimension

- Let a_X, a_Y be vectorized versions of A_X, A_Y
  - Stack rows on top of each other
- $X \cong Y \iff PA_XP^{-1} = A_Y \iff P^{\otimes 2}a_X = a_Y$

Naturally extends to higher dimensions $n > 2$:

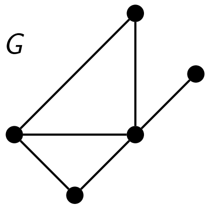
- $A_X : V(X)^2 \to \{0,1\}$ and $A_Y : V(Y)^2 \to \{0,1\}$
  - Functions on $n = 2$ inputs.
- $F : V(F)^n \to \mathbb{R}$ and $G : V(G)^n \to \mathbb{R}$ – **constraint functions**.
  - Vectorize as f, g.
- $F \cong G$ if there is a permutation matrix P s.t. $P^{\otimes n}f = g$.
- $F \cong_{qc} G$ if there is a quantum permutation matrix $\mathcal{U}$ s.t. $\mathcal{U}^{\otimes n}f = g$.



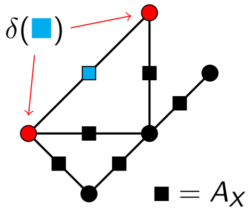


Recall: $\hom(G, X)$ is the number of maps $\phi : V(G) \to V(X)$ that send every edge of G to an edge of X.

The partition function



Recall: hom(G, X) is the number of maps $\phi : V(G) \to V(X)$ that send every edge of G to an edge of X.



For each $\blacksquare$, let $\delta(\blacksquare)$ be the two adjacent $\bullet$ vertices.

Then $A_X(\phi(\delta(\blacksquare))) = 1$ if ϕ maps $\blacksquare$'s edge to an edge of X, and $= 0$ otherwise.

$$\hom(G, X) = \sum_{\phi:\{\bullet\}\to V(X)} \prod_{\blacksquare} A_X(\phi(\delta(\blacksquare))).$$

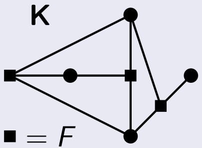
(product is 1 iff ϕ maps every edge of G to an edge of X).

$\blacksquare$ are **constraints** $\bullet$ are **variables**

#CSP: higher-dimensional graph homomorphism

- A_X 2-dimensional, so ■ vertices have degree 2 ($|\delta(\blacksquare)| = 2$)
- For n-dimensional F, ■ vertices have degree n ($|\delta(\blacksquare)| = n$).
- *Signature grid* **K**.

Example: $n = 3$



$$\mathrm{hom}(G, X) = \sum_{\phi:\{\bullet\}\to V(X)} \prod_{\blacksquare} A_X(\phi(\delta(\blacksquare))).$$

$$\#\mathrm{CSP}(\mathbf{K}, F) = \sum_{\phi:\{\bullet\}\to V(F)} \prod_{\blacksquare} F(\phi(\delta(\blacksquare))).$$

Theorem (Mančinska-Roberson'20)

$X \cong_{qc} Y$ *iff* $\mathrm{hom}(G, X) = \mathrm{hom}(G, Y)$ $\forall$ *planar graph* G.

Theorem (Cai-Y.'23)

$F \cong_{qc} G$ *iff* $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)$ $\forall$ *planar signature grid* $\mathbf{K}$.

Why study (planar) #CSP?

- Beautiful complexity dichotomy theorems

Theorem (Cai-Chen'17)

For any finite set $\mathcal{F}$ of $\mathbb{C}$-valued constraint functions, $\#\mathrm{CSP}(\cdot, \mathcal{F})$ is always either in P or #P-hard, with nothing in between.

Theorem (Cai-Fu'19)

For any set $\mathcal{F}$ of $\mathbb{C}$-valued constraint functions over Boolean variables, $\#\mathrm{CSP}(\cdot, \mathcal{F})$ is exactly one of the following:

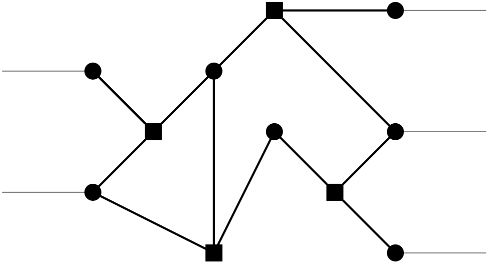
1. *P-time solvable;*
2. *P-time solvable over planar graphs but #P-hard over general graphs;*
3. *#P-hard over planar graphs.*

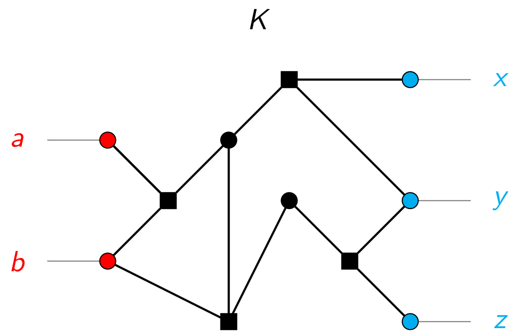
Theorem (Cai-Y.'23)

$F \cong_{qc} G$ *iff* $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)$ $\forall$ *planar signature grid* $\mathbf{K}$.

Gadgets

- A **gadget** is a signature grid with **dangling edges**.
- Several constraint functions assembled into a new function.
- Inputs along dangling edges.



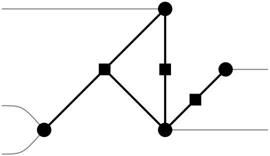


- $a, b, x, y, z \in V(F)$
- $|V(F)|^2 \times |V(F)|^3$ signature matrix $M(\mathbf{K})$.

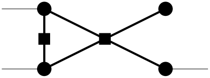
$$M(\mathbf{K})_{ab,xyz} = \sum_{\substack{\phi:\{\bullet\}\to V(F) \\ \phi(\bullet,\bullet)=(a,b) \\ \phi(\bullet,\bullet,\bullet)=(x,y,z)}} \prod_{\blacksquare} F(\phi(\delta(\blacksquare))).$$

Gadget operations

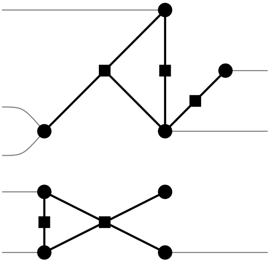
K



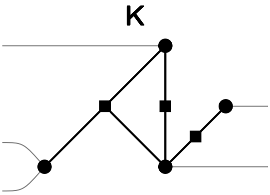
L

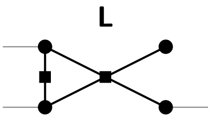


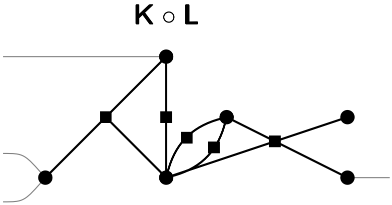
K $\otimes$ **L**



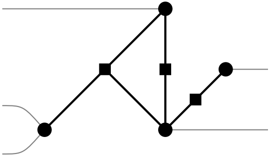
Gadget operations



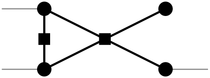




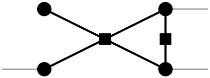
K



L



$L^\dagger$

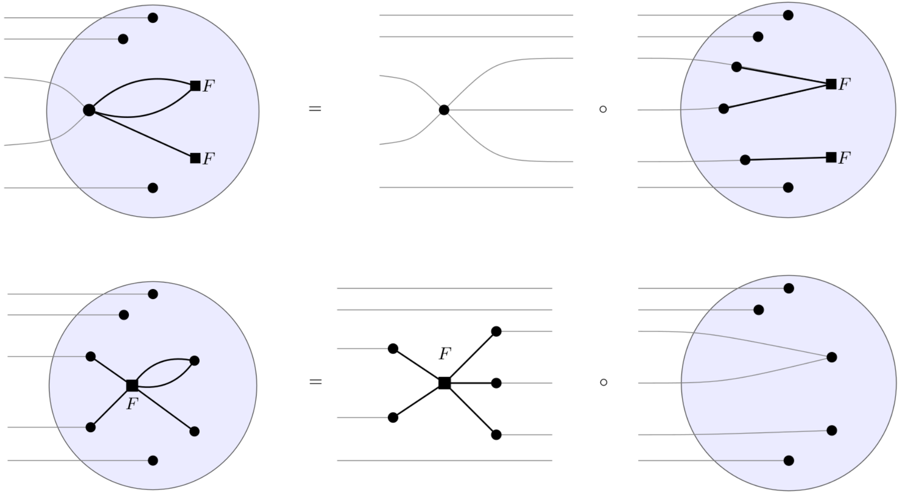


Gadgets operations

- Gadget operations correspond to signature matrix operations:
  - $M(\mathbf{K} \otimes \mathbf{L}) = M(\mathbf{K}) \otimes M(\mathbf{L})$
  - $M(\mathbf{K} \circ \mathbf{L}) = M(\mathbf{K})M(\mathbf{L})$
  - $M(\mathbf{K}^\dagger) = M(\mathbf{K})^\dagger$

Why planarity? – the planar gadget decomposition

- Can decompose any planar gadget into a chain of simple gadgets



A quantum holographic transformation

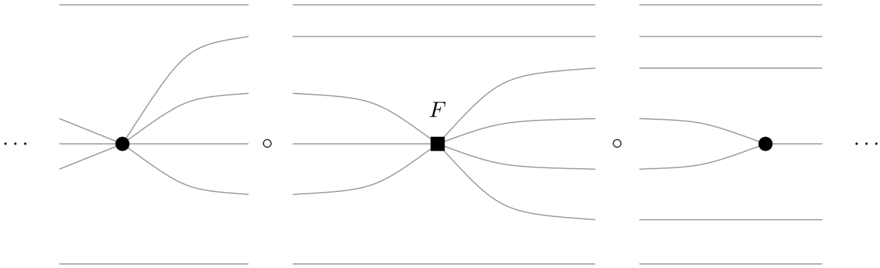
Recall our main theorem:

Theorem (Cai-Y.'23)

$F \cong_{qc} G$ *iff* $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)\ \forall$ *planar* $\mathbf{K}$.

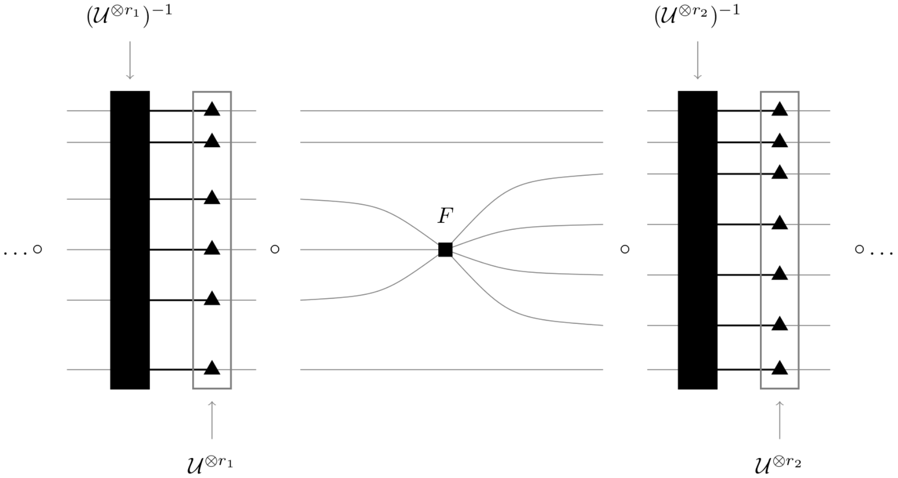
- Suppose $F \cong_{qc} G$, so $\mathcal{U}^{\otimes n} f = g$ for quantum permutation matrix $\mathcal{U}$.
- Prove ($\implies$) via a **holographic transformation** using $\mathcal{U}$.
- View $\mathcal{U}$ itself as a constraint function in the signature grid.

A quantum holographic transformation

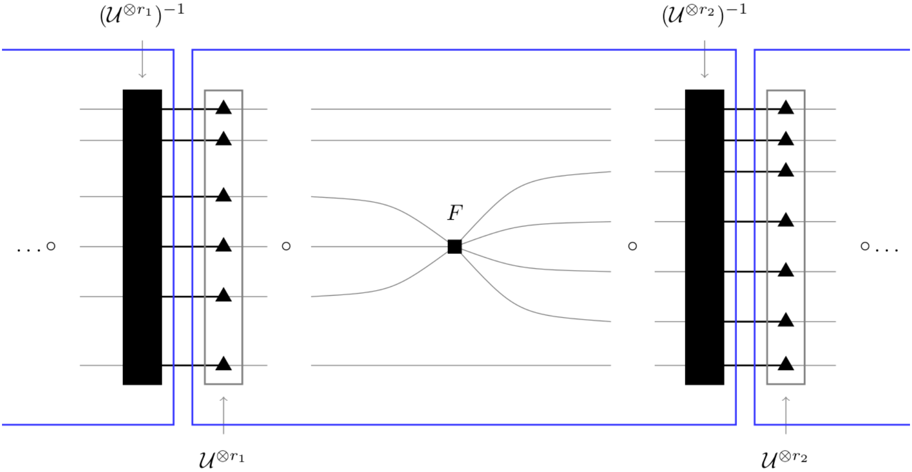


The planar gadget decomposition converts **K** to a composition of building block gadgets (expressing $\#\text{CSP}(\mathbf{K}, F)$ as a matrix product).

A quantum holographic transformation

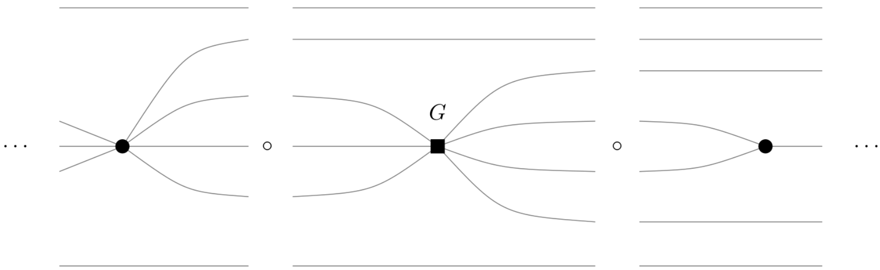


Insert $(\mathcal{U}^{\otimes r_i})^{-1}\mathcal{U}^{\otimes r_i} = I$ between the ith and $(i+1)$st factors (preserves $\#\mathrm{CSP}(\mathbf{K}, F)$ value).



Reassociate. Now, $\mathcal{U}^{\otimes n} f = g \iff \mathcal{U}^{\otimes m} M(F)(\mathcal{U}^{\otimes n-m})^{-1} = M(G)$ and $\mathcal{U}$ doesn't affect • vertices, so...

A quantum holographic transformation



Every F is converted to G without changing the #CSP value:
$\#\mathrm{CSP}(\mathbf{K}, F) = \#\mathrm{CSP}(\mathbf{K}, G)$.

A quantum holographic transformation

- Can't view $\mathcal{U}$ as a constraint function in general (nonplanar) signature grids because entries of $\mathcal{U}$ don't commute.
  - #CSP value is a sum of products of constraint function evaluations.
- Planar gadget decomposition gives order of vertices
  - hence multiplication order of $\mathcal{U}$ entries.

Theorem (Cai-Y.'23)

$F \cong_{qc} G$ iff $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)$ $\forall$ planar $\mathbf{K}$.

- Next, prove ($\Leftarrow$). Apply quantum group theory:

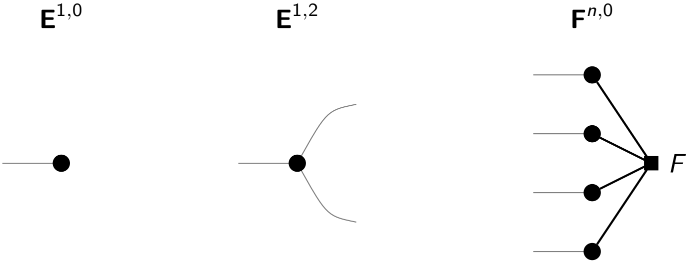
Definition

Quantum permutation matrix $\mathcal{U}$ s.t. $\mathcal{U}^{\otimes n} f = f$ defines the *quantum automorphism group* $\text{Qut}(F)$ of F.

- Recall $\mathcal{U}^{\otimes n} f = g$ defined quantum isomorphism of F and G.
- Instead of studying $\text{Qut}(F)$ directly, study its **intertwiner space** of matrices M invariant under $\mathcal{U}$.
  - $\mathcal{U}^{\otimes m} M (\mathcal{U}^{\otimes d})^{-1} = M$
- F itself is invariant under $\mathcal{U}$.
  - Suggests that $\text{Qut}(F)$'s intertwiner space is composed of $\#\text{CSP}(\cdot, F)$ gadget signature matrices.

Theorem

The set of all planar $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0} \rangle_{\circ,\otimes,\dagger}$.



Follows from the earlier planar gadget decomposition.

Characterization of the intertwiners

Proved combinatorially:

Theorem

The set of all planar $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0} \rangle_{\circ,\otimes,\dagger}$.

Proved using quantum group theory:

Theorem

The intertwiners of $\mathrm{Qut}(F)$ are exactly

$$\mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}) \rangle_{\circ,\otimes,\dagger}).$$

$$\text{intertwiners of } \mathrm{Qut}(F)$$

$$\updownarrow$$

$$\text{span(signature matrices of planar } \#\mathrm{CSP}(\cdot, F) \text{ gadgets)}$$

The converse

- Connection between Qut(F) and gadget signature matrices $\rightsquigarrow$ more quantum group theory!
- Analogues of techniques from classical graph isomorphism:
- Orbits of Qut(F)
- Make F and G ‘connected’ by adding a ‘universal vertex’.
- Disjoint union $F \oplus G$ of two constraint functions F and G.
- If there is some $x \in V(F)$, $y \in V(G)$ in the same orbit of Qut($F \oplus G$), then $F \cong_{qc} G$.

Future work

- $\#\mathrm{CSP}(\cdot, \mathcal{F}) \leftrightarrow \mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ})$
- For quantum **orthogonal** $\mathcal{O}$: $\mathcal{O} \cdot \mathcal{EQ} = \mathcal{EQ} \iff \mathcal{O}$ is a quantum permutation matrix.
- (orthogonal) Holant theorem: for orthogonal H,

$$\mathrm{Holant}_\Omega(H\,\mathcal{F}) = \mathrm{Holant}_\Omega(\mathcal{F}) \text{ on every signature grid } \Omega$$

- Our proof shows: for **quantum** orthogonal $\mathcal{O}$,

$$\mathrm{Holant}_\Omega(\mathcal{O}\,\mathcal{F}) = \mathrm{Holant}_\Omega(\mathcal{F}) \text{ on every } \textbf{planar} \text{ signature grid } \Omega$$

Conjecture

If $\mathrm{Holant}_\Omega(\mathcal{F}) = \mathrm{Holant}_\Omega(\mathcal{G})$ on every planar signature grid Ω, then there is a quantum orthogonal $\mathcal{O}$ s.t. $\mathcal{O}\,\mathcal{F} = \mathcal{G}$

Thank you!
Questions?