Indistinguishability: Counting, Constraints, and Contractions

Benjamin Morgen Young

A dissertation defense in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy
(Computer Sciences)
at the
UNIVERSITY OF WISCONSIN-MADISON
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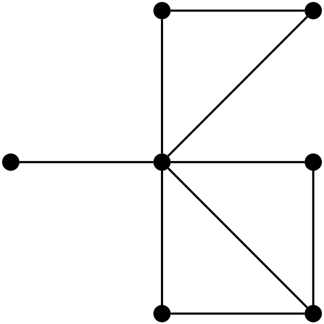
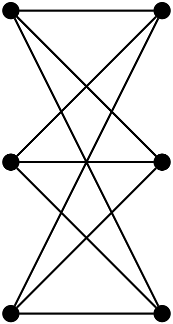
Committee:
Jin-Yi Cai   Eric Bach   Rishab Goyal   Paul Terwilliger

Graphs





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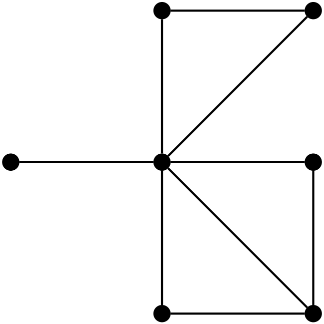
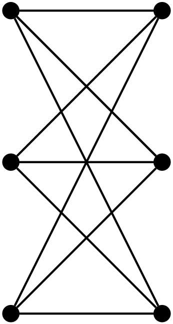


Planar



Not planar

Signatures

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  - **Domain** $[q] := \{0, 1, \ldots, q-1\}$
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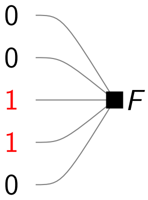

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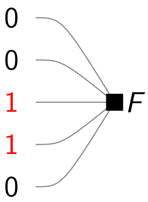

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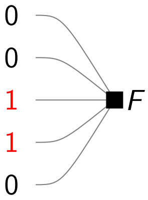

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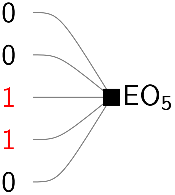

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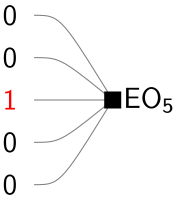

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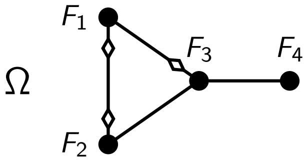

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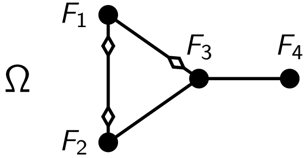
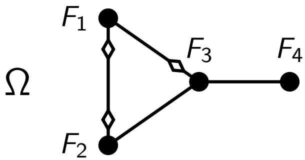
- $\mathcal{F}$-**grid** (tensor network) Ω is a graph with a signature from $\mathcal{F}$ on each vertex.
  - Arity of signature equals degree of vertex.
  - Order incident edges counterclockwise starting from diamond.



Example: Counting Perfect Matchings



- Goal: compute the **Holant value** (contraction) of Ω:

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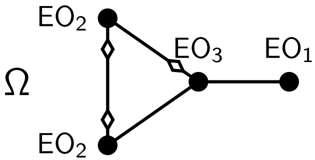

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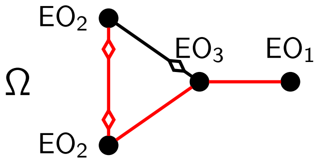

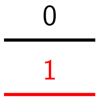

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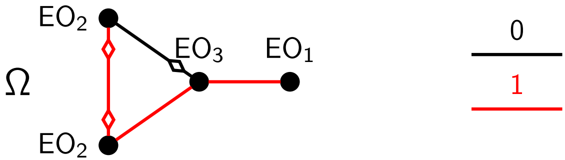

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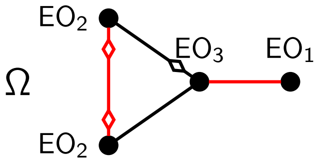

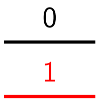

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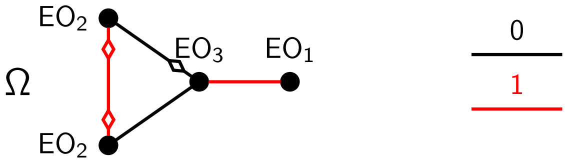

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Broad dichotomies exist for $\mathcal{F}$ containing signatures that are

- Domain $q = 2$, $\mathbb{C}$-valued, symmetric [Cai-Guo-Williams'16]
- Domain $q = 2$, $\mathbb{R}$-valued [Shao-Cai'20]
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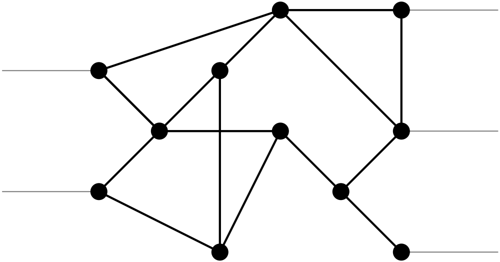
- Our techniques apply to any q.

Gadgets

- A **gadget** is a signature grid with **dangling edges**.
- Here, signatures assembled into a 5-ary signature M.



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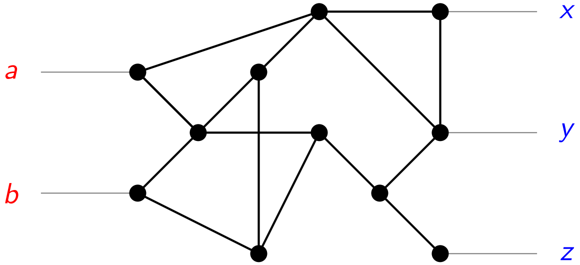

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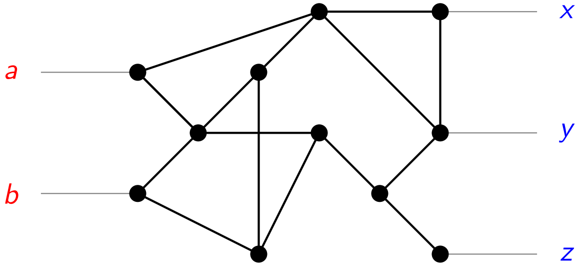

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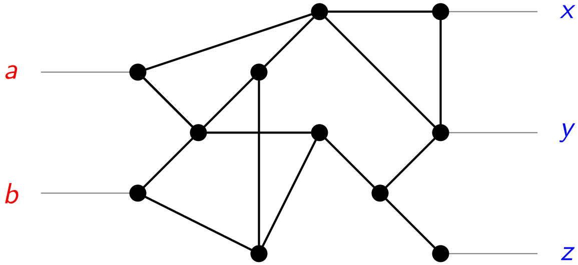
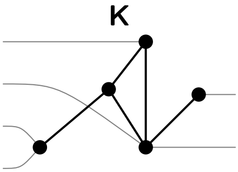
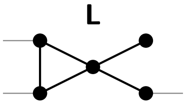
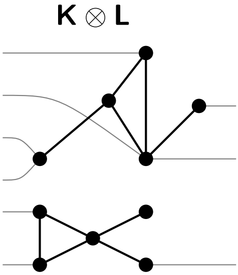


- $a, b, x, y, z \in [q]$.
- Organize signature in $q^2 \times q^3$ matrix M.
- $M(ab, xyz)$ is the Holant value with dangling edges fixed to a, b, x, y, z.

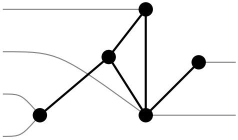
Gadget Operations



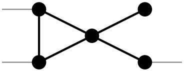




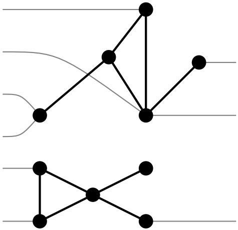
K



L



$\mathbf{K} \otimes \mathbf{L}$



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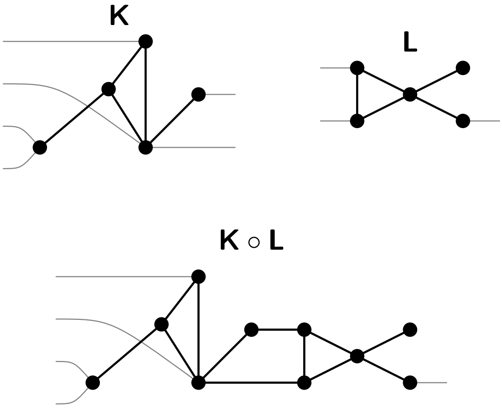
K
L
K ∘ L

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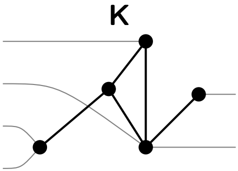

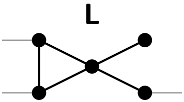

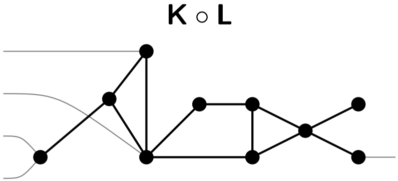

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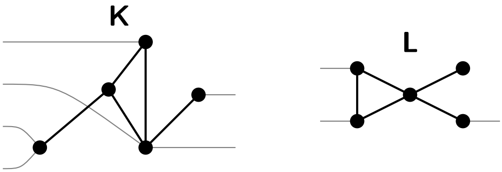
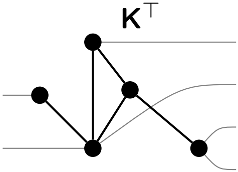
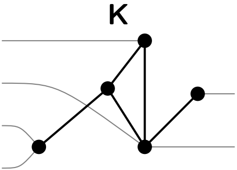
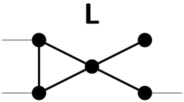
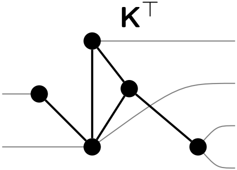




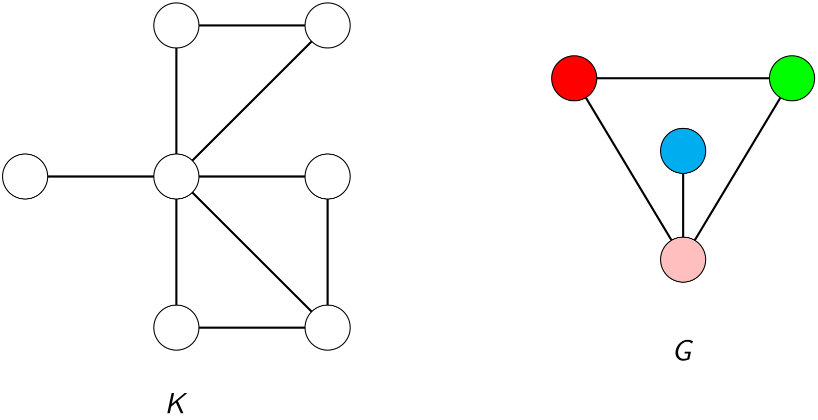






- $M(\mathbf{K}^\top) = M(\mathbf{K})^\top$

A mapping $\phi : V(K) \to V(G)$ is a **graph homomorphism** if it maps all edges to edges: $\{u, v\} \in E(K) \implies \{\phi(u), \phi(v)\} \in E(G)$.



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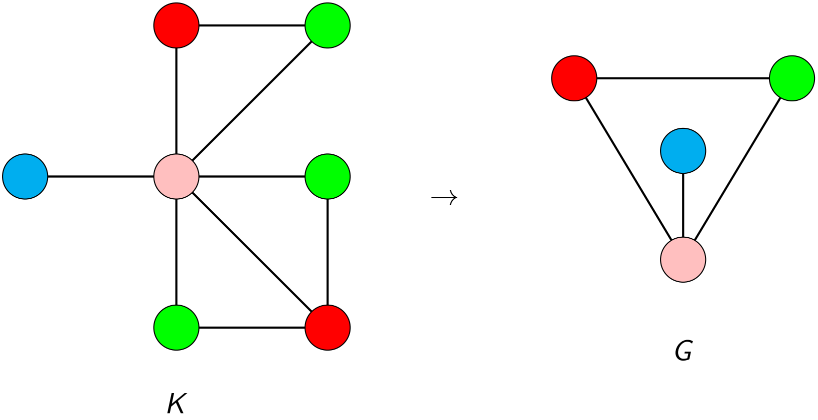

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Theorem ([Mančinska-Roberson'20])

X and Y are homomorphism-indistinguishable over planar graphs iff X and Y are quantum isomorphic.

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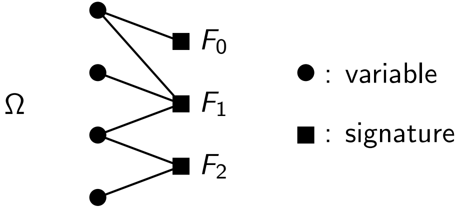
- Get relaxations of isomorphism for other $\mathfrak{G}$:
  - Known for planar, bipartite, cycles, bounded tree/pathwidth, etc.
  - Was open for graphs of degree $\leq d$

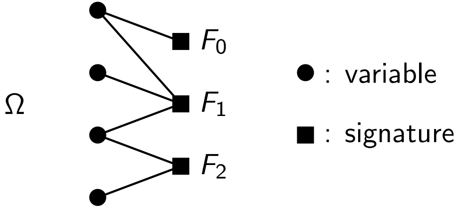
Theorem ([Mančinska-Roberson'20])

X and Y are homomorphism-indistinguishable over planar graphs iff X and Y are quantum isomorphic.

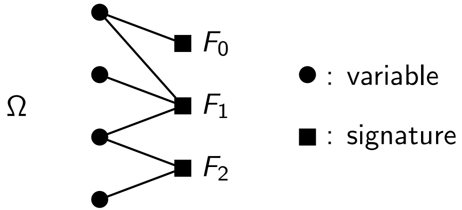
- First goal: extend these two theorems to #CSP.

#CSP: the Counting Constraint Satisfaction Problem





$$\#\mathrm{CSP}(\Omega, \mathcal{F}) = \sum_{\sigma:\{\text{variables}\}\to[q]} \prod_{\text{signature } F} F(\sigma(\text{incident variables})).$$



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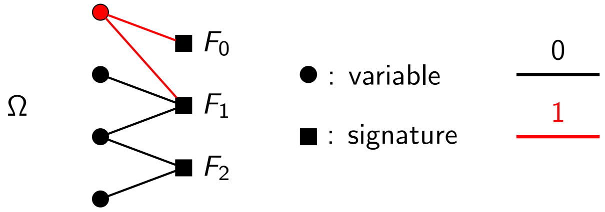
- Example with $q = 2$ and $F_i(\boldsymbol{x}) = 1$ iff x has at least one 1.



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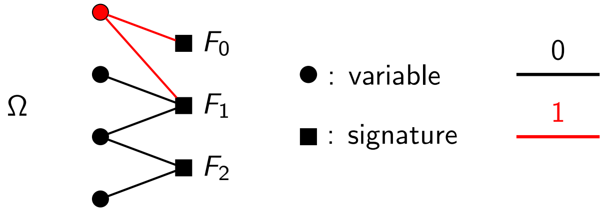

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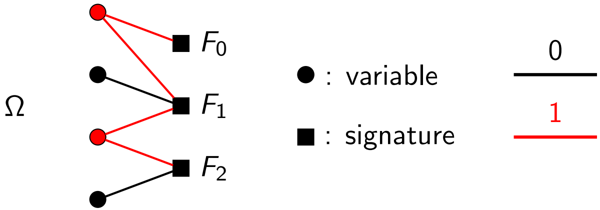

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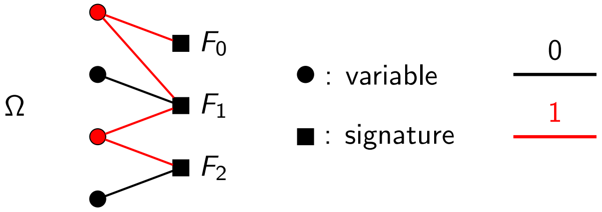

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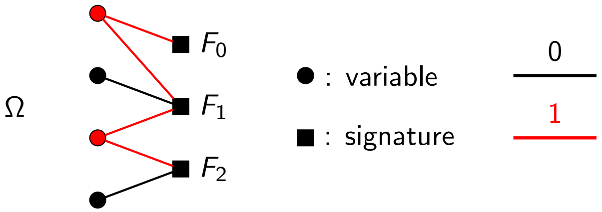
#CSP as a Holant problem



- $\mathcal{EQ} = \{=_n \mid n \in \mathbb{N}\}$ where

$$(=_n)(x_1, \ldots, x_n) = \begin{cases} 1 & x_1 = \ldots = x_n \\ 0 & \text{otherwise} \end{cases}$$

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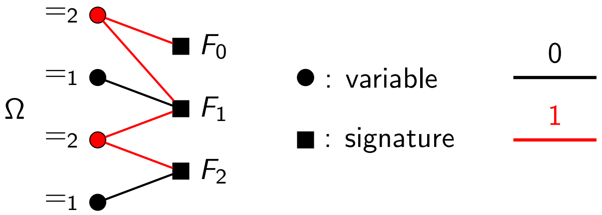

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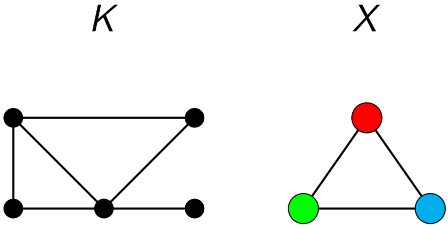
- $\#\mathrm{CSP}(\mathcal{F}) \equiv \mathrm{Holant}(\mathcal{EQ} \cup \mathcal{F})$.

#CSP: Counting Graph Homomorphisms

- $\#\text{hom}(\cdot, X)$



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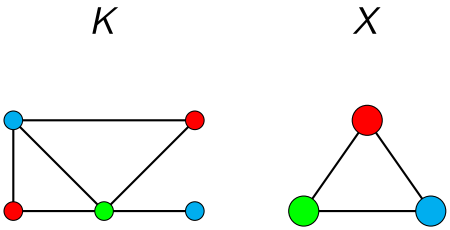

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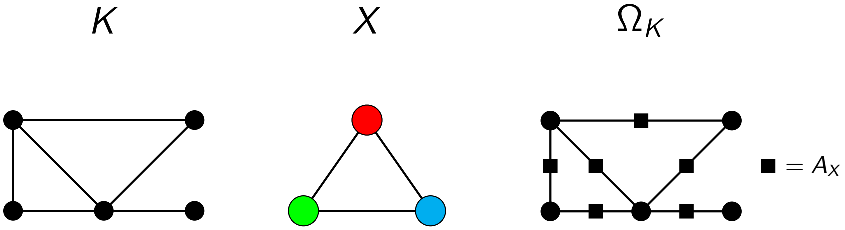

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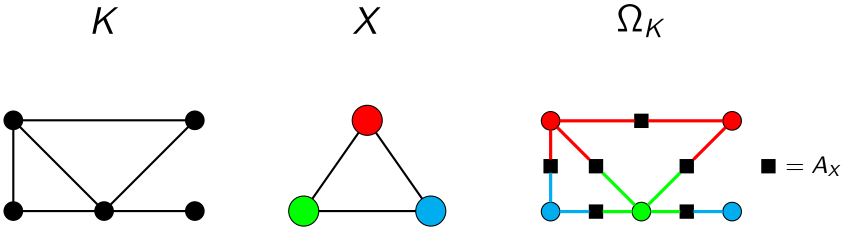

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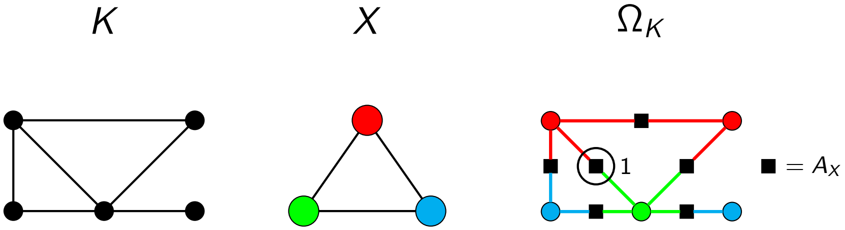

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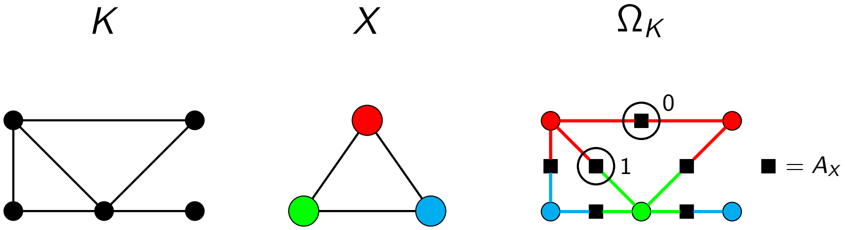

Why Study (planar) #CSP?

Theorem (Cai-Fu'19)

For any set $\mathcal{F}$ of $\mathbb{C}$-valued signatures on domain $q = 2$, $\#$CSP$(\cdot, \mathcal{F})$ is exactly one of the following:

1. *P-time solvable;*
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Theorem ([Cai-Maran'23; Cai-Maran'24])

For any graph X with nonnegative edge weights and $q = 3, 4$ vertices, #hom$(\cdot, X)$ is exactly one of the following:

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- What is quantum isomorphism?

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- Alice and Bob try to convince a Verifier that graphs $X \cong Y$.

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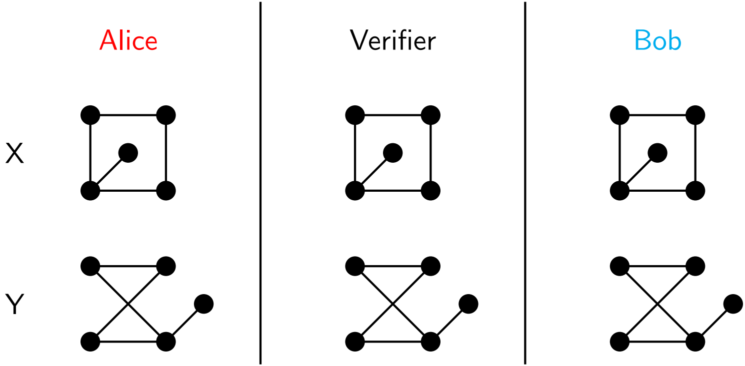

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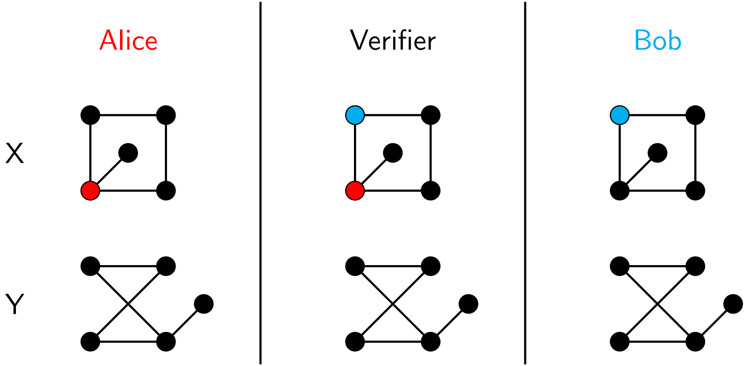

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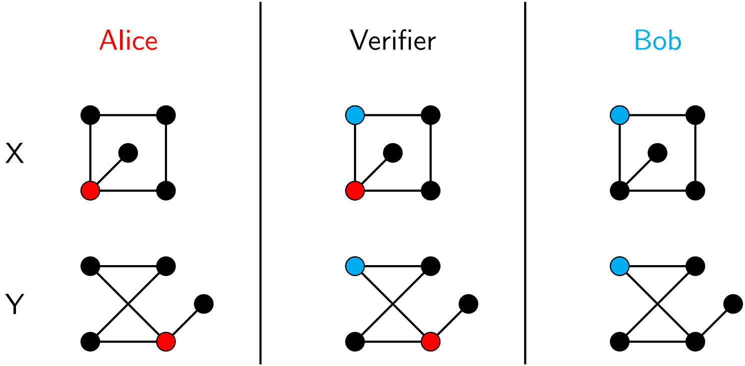

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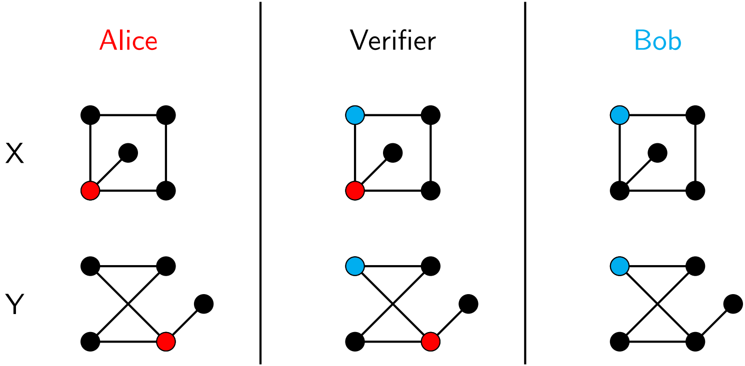

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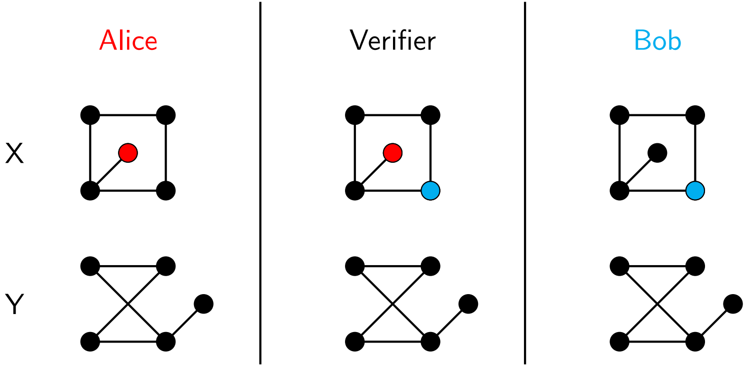

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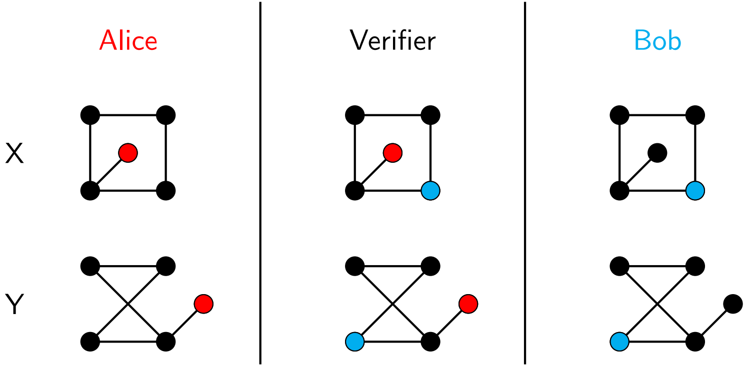

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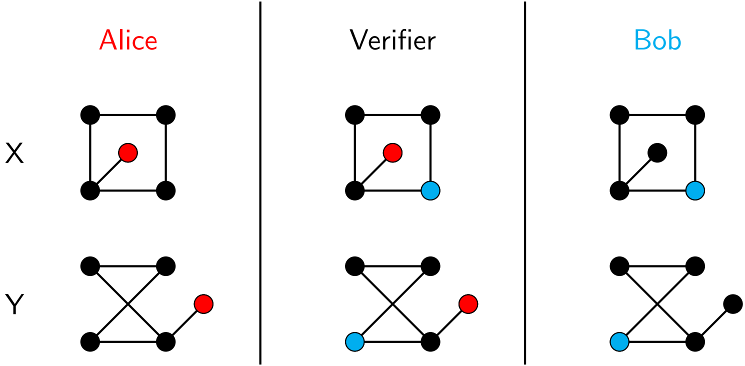

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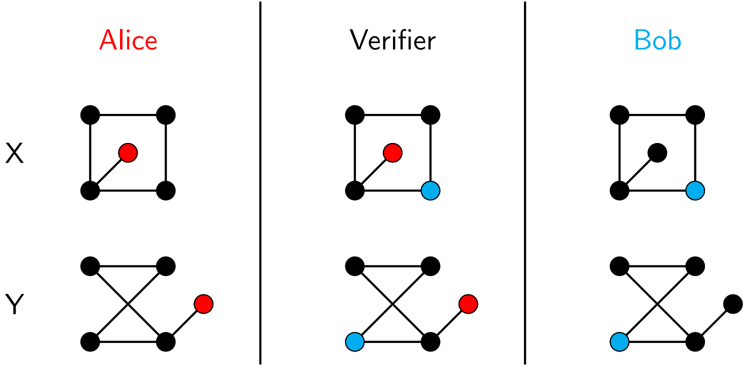

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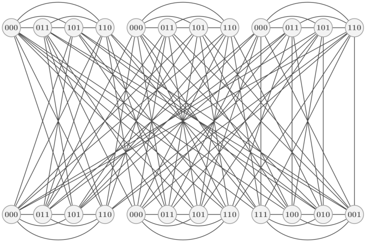

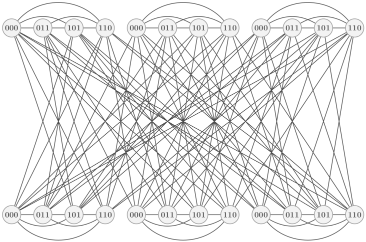

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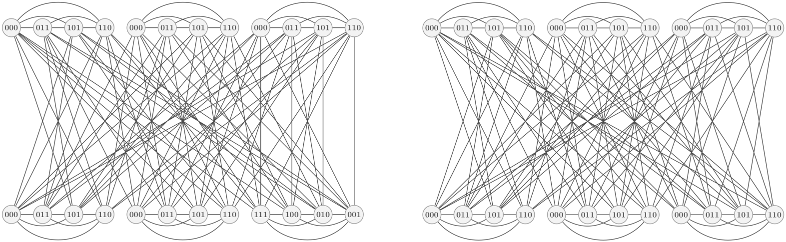

- Quantum isomorphism is **undecidable** [AMRSSV'19, Slofstra'19]

Theorem (AMRSSV'18, Lupini-Mančinska-Roberson'17)

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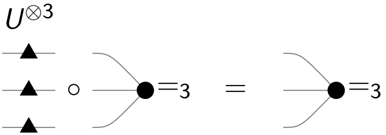

$\iff$ $U_{ij} U_{ij'} = \delta_{jj'} U_{ij}$

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Quantum Isomorphism of Constraint Functions

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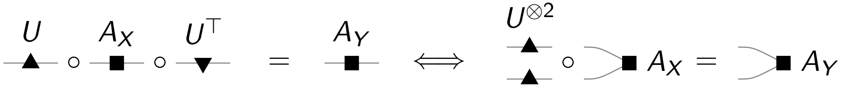

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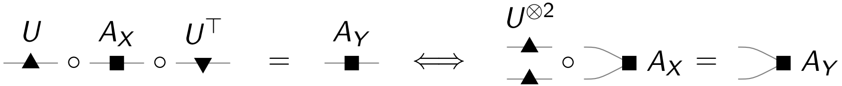
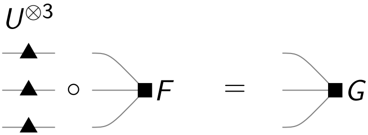


Definition

$F \cong_q G$ if there is a quantum permutation matrix U such that $U^{\otimes n} F = G$.



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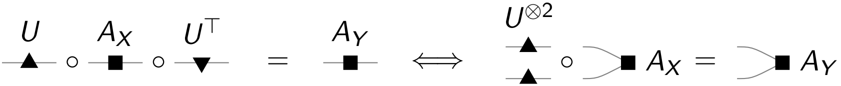

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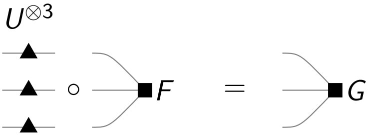

- $\mathcal{F} \cong_q \mathcal{G}$ if there is a U such that $U\mathcal{F} = \mathcal{G}$ (simultaneous action).

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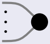
- First consider 'easy' direction ($\Longleftarrow$).

Theorem (The planar gadget decomposition)

$\Big\langle \Big\{$  $F \Big\}_{F \in \mathcal{F}}$,  $\Big\rangle_{\circ, \otimes, \top}$ = *all* Pl-Holant($\mathcal{F}$) *gadgets*

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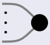
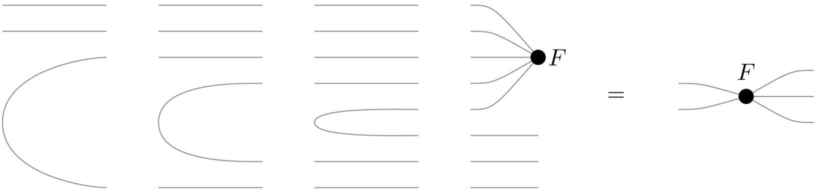
$\Big\langle \Big\{$  $F \Big\}_{F \in \mathcal{F}}$,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all* PI-Holant($\mathcal{F}$) *gadgets*



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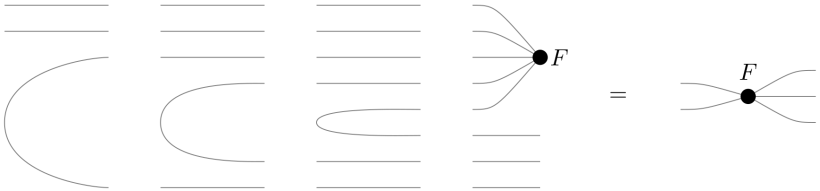

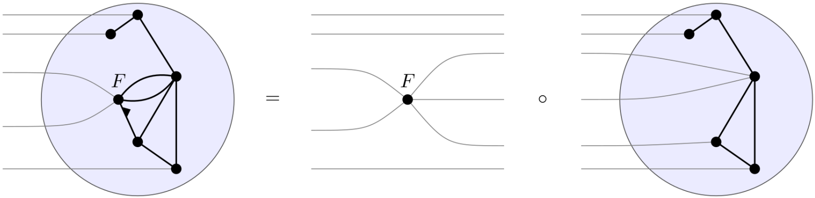

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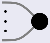
$\Big\langle \Big\{$  $F \Big\}_{F \in \mathcal{F}}$,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all* PI-Holant($\mathcal{F}$) *gadgets*

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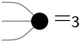
- $\mathcal{EQ} \subset \Big\langle$ 

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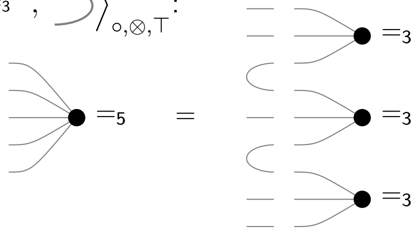

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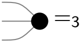
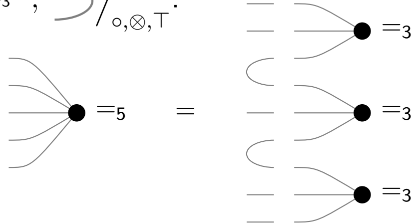
- $\mathcal{EQ} \subset \Big\langle$  ,  $\Big\rangle_{\circ, \otimes, \top}$:



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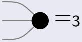
$\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all planar* $\#\text{CSP}(\mathcal{F})$ *gadgets*

Quantum isomorphism implies planar indistinguishability

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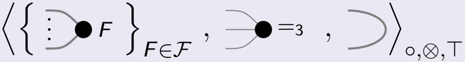

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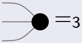
$\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  $\Big\rangle_{\circ, \otimes, \top} =$ *all* PI-#CSP($\mathcal{F}$) *gadgets*

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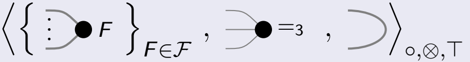

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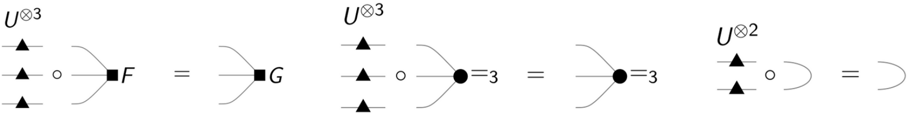

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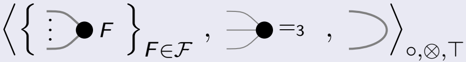

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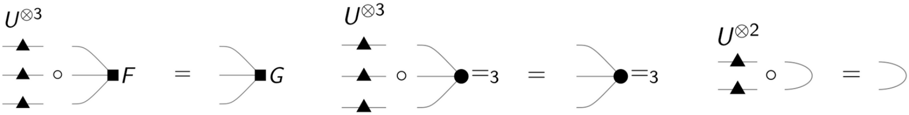

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If $\mathcal{F} \cong_q \mathcal{G}$*, then* $\mathcal{F}$ *and* $\mathcal{G}$ *are* PI-#CSP*-indistinguishable.*

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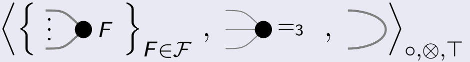

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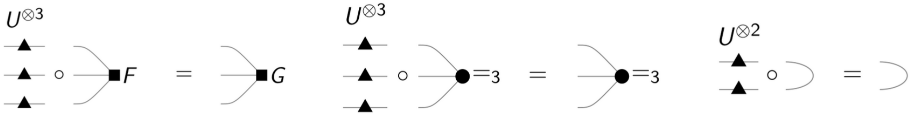

Corollary

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- Next, show the converse ('hard' direction)

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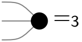
- $\mathcal{F}$, 

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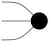
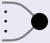
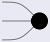
- $\mathcal{F}$,  $=_3$,  invariant under $\mathrm{Qut}(\mathcal{F})$.

Corollary (Planar #CSP gadget decomposition)

$\Big\langle \Big\{$  $F \Big\}_{F\in\mathcal{F}}$,  $=_3$,  $\Big\rangle_{\circ,\otimes,\top}$ = *all* PI-#CSP($\mathcal{F}$) *gadgets*

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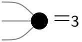
- $\mathcal{F}$, 

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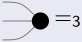
$\Big\}_{F \in \mathcal{F}}$, 

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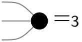
- $\mathcal{F}$,  ,  invariant under $\mathrm{Qut}(\mathcal{F})$.

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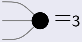
$\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  $\Big\rangle_{\circ, \otimes, \top}$ = *all* PI-#CSP($\mathcal{F}$) *gadgets*

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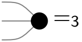
- $\mathcal{F}$,  ,  invariant under $\mathrm{Qut}(\mathcal{F})$.

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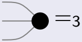
$\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  $\Big\rangle_{\circ,\otimes,\top}$ = *all* PI-#CSP($\mathcal{F}$) *gadgets*

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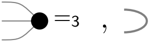
- $\mathcal{F}$,  invariant under $\mathrm{Qut}(\mathcal{F})$.

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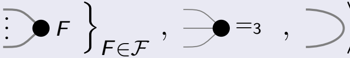
$\Big\langle \Big\{$  $\Big\}_{F\in\mathcal{F}}$, … $\Big\rangle_{\circ,\otimes,\top}$ = *all* PI-#CSP($\mathcal{F}$) *gadgets*

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Theorem ([Cai-Y.'23])

$\mathcal{F}$ and $\mathcal{G}$ are PI-#CSP*-indistinguishable iff $\mathcal{F} \cong_q \mathcal{G}$.*

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Corollary ([Cai-Maran-Y.'26])

Undecidable: given X, x, y, is there a binary PI-#hom$(\cdot, X)$*-gadget* $\mathbf{K}$ *such that $M(\mathbf{K})_{xx} \neq M(\mathbf{K})_{yy}$?*

Nonplanarity: the swap gadget

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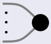
$\Big\langle \Big\{$  F $\Big\}_{F \in \mathcal{F}}$,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all* Pl-Holant($\mathcal{F}$) *gadgets*

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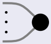
$\Big\langle \Big\{$  F $\Big\}_{F \in \mathcal{F}}$,  ,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all* Holant($\mathcal{F}$) *gadgets*

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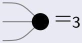
$\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  ,  $\Big\rangle_{\circ, \otimes, \top}$ $=$ *all* #CSP($\mathcal{F}$) *gadgets*

- Recall: if $\mathcal{F} = U\,\mathcal{G}$ and the entries of U commute, then $\mathcal{F} \cong \mathcal{G}$.

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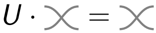
-  $\iff$ entries of U commute.

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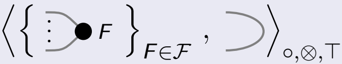

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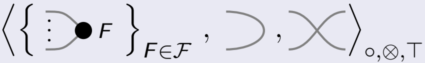

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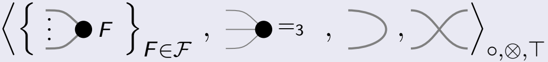

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Corollary ([Y.'25a])

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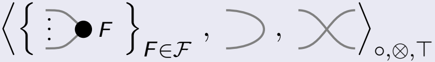

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The orthogonal Holant theorem

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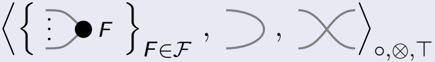

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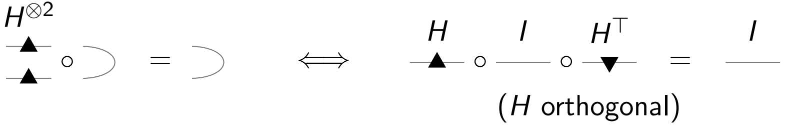
- Let H be a 'normal' matrix. Recall:



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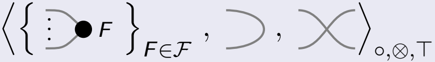

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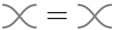
(H orthogonal)

- $H \cdot$  because entries of H commute.

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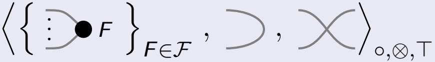

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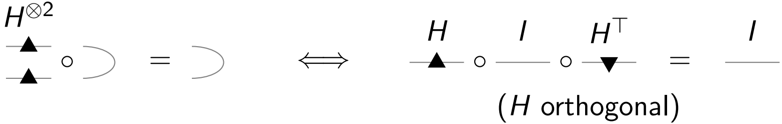

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Corollary (The orthogonal Holant theorem)

If $\mathcal{F} = H\mathcal{G}$ for orthogonal H, then $\mathcal{F}$ and $\mathcal{G}$ are Holant-*indistinguishable.*

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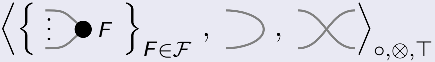

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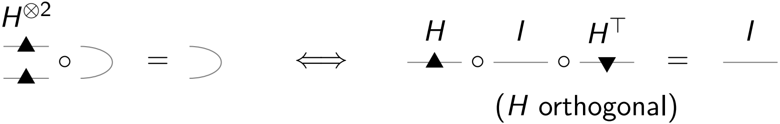
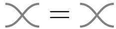


- $H \cdot$  because entries of H commute.

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Theorem ([Y.’25b])

$\mathcal{F} = H\mathcal{G}$ *for orthogonal* H *iff* $\mathcal{F}$ *and* $\mathcal{G}$ *are* Holant-*indistinguishable.*

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- $\mathcal{F}\,|_X$ and $\mathcal{G}\,|_X$ are Holant-indistinguishable if $\mathcal{F}, \mathcal{G}$ contain $\mathbb{1}_X$:

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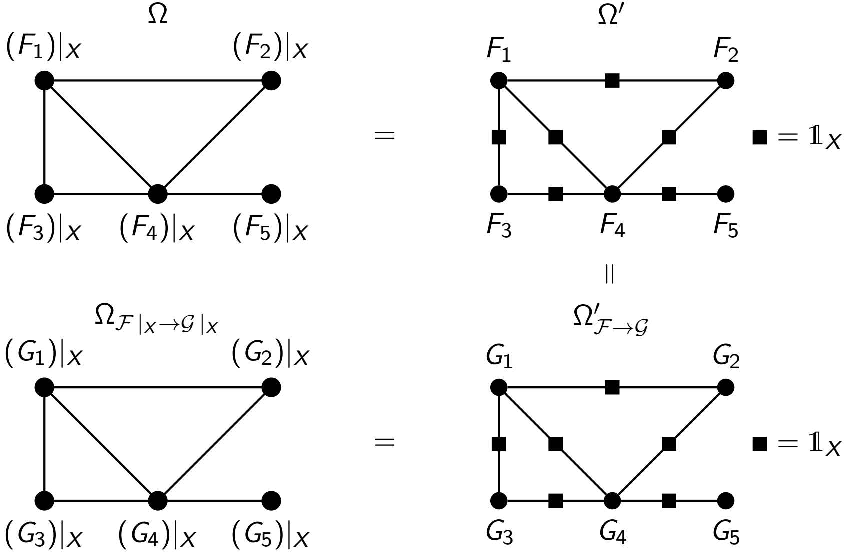

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- How to obtain D?

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Lemma (consequence of Tannaka-Krein duality)

$M(\mathbf{K})_x = M(\mathbf{K})_y$ *for every* PI-#CSP$(\mathcal{F} \oplus \mathcal{G})$*-gadget* $\mathbf{K}$ *with one dangling edge iff* x *and* y *are in the same orbit of* $\mathrm{Qut}(\mathcal{F} \oplus \mathcal{G})$.

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- i.e. $H|_{\mathcal{G},\mathcal{F}} \neq 0$ or $H|_{\mathcal{F},\mathcal{G}} \neq 0$.

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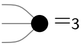
- Apply SVD to $H_{V(\mathcal{G}),V(\mathcal{F})}$ to obtain D for inductive lemma.

From planar to bipartite

- $\Big\langle \Big\{$  $\Big\}_{F \in \mathcal{F}}$,  ,  ,  $\Big\rangle_{\circ, \otimes, \top}$ $= \#\mathrm{CSP}(\mathcal{F})$ gadgets
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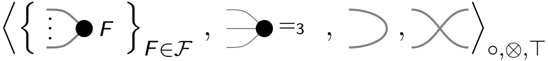
- 

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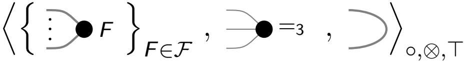
- 

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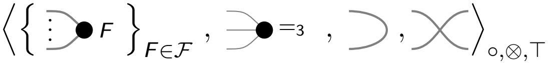
-  $= \#\mathrm{CSP}(\mathcal{F})$ gadgets
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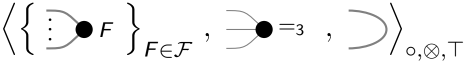
-  $=$ PI-$\#\mathrm{CSP}(\mathcal{F})$ gadgets
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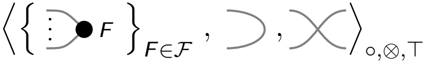
-  $= \mathrm{Holant}(\mathcal{F})$ gadgets
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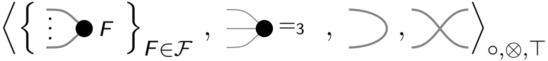
-  $= \#\mathrm{CSP}(\mathcal{F})$ gadgets
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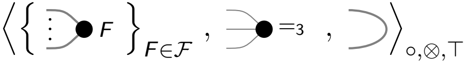
-  $=$ Pl-$\#\mathrm{CSP}(\mathcal{F})$ gadgets
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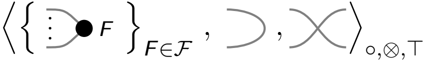
-  $= \mathrm{Holant}(\mathcal{F})$ gadgets
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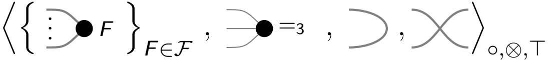
- 

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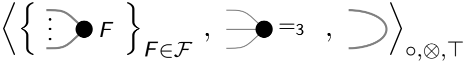
- 

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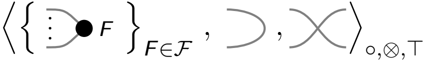
- 

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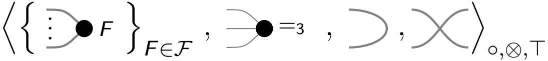
- 

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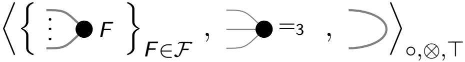
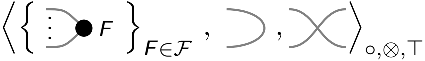
- 

$=$ PI-$\#\mathrm{CSP}(\mathcal{F})$ gadgets
  - quantum isomorphism
- 

$=$ Holant$(\mathcal{F})$ gadgets
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- 

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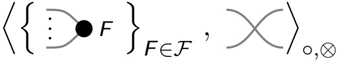
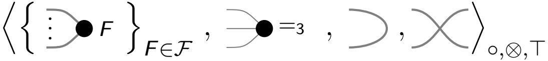
- 

$\approx$ Bi-Holant$(\mathcal{F})$ gadgets

From planar to bipartite

-  $= \#\mathsf{CSP}(\mathcal{F})$ gadgets
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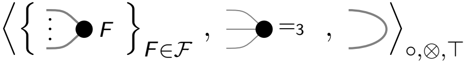
-  $= \mathsf{Pl}\text{-}\#\mathsf{CSP}(\mathcal{F})$ gadgets
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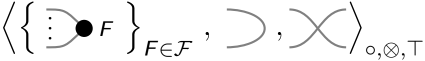
-  $= \mathsf{Holant}(\mathcal{F})$ gadgets
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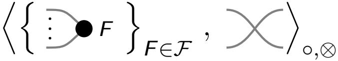
-  $\approx \mathsf{Bi}\text{-}\mathsf{Holant}(\mathcal{F})$ gadgets
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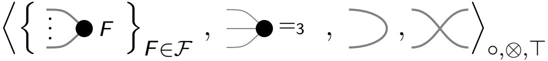
-  $= \#\text{CSP}(\mathcal{F})$ gadgets
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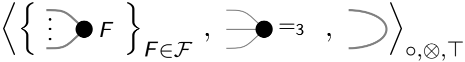
-  $=$ PI-$\#\text{CSP}(\mathcal{F})$ gadgets
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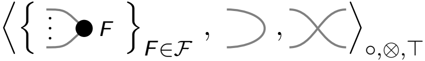
-  $=$ Holant($\mathcal{F}$) gadgets
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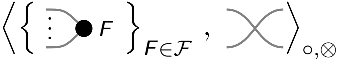
-  $\approx$ Bi-Holant($\mathcal{F}$) gadgets
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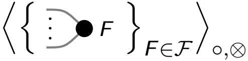
-  $=$??? gadgets
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- What does  do?

Bipartite Holant

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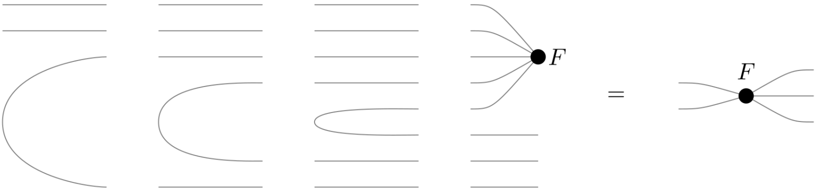

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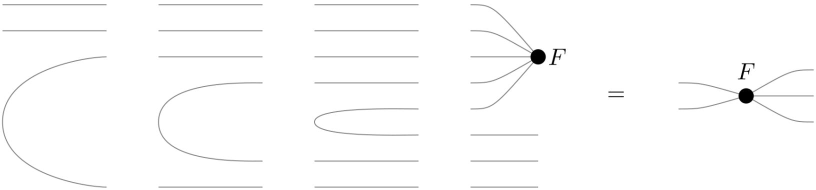
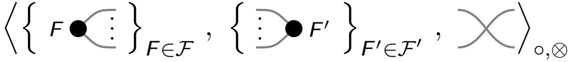


- ⊤ also switches dangling edges between left and right.

Bipartite Holant



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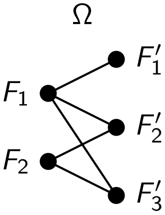

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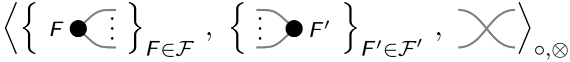

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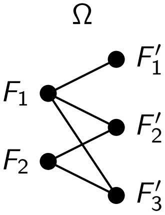

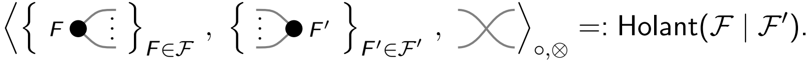

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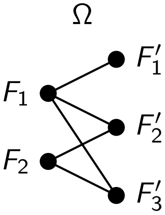

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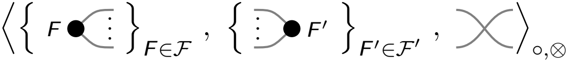

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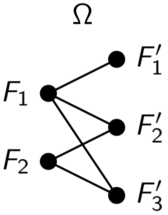

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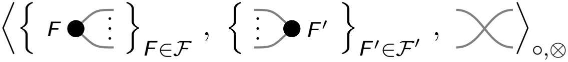

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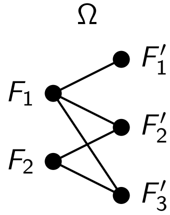

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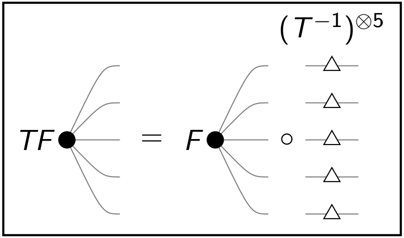

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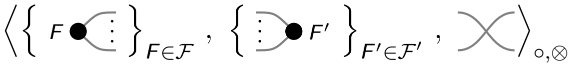

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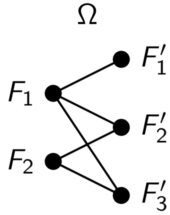

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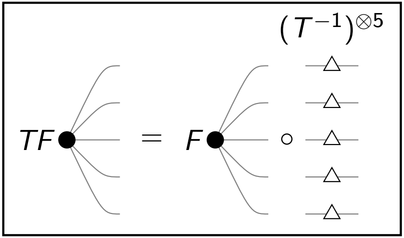

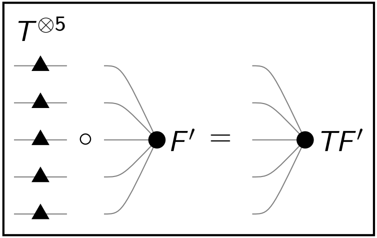

The Holant Theorem

Theorem (The Holant Theorem [Valiant'08])

If $\mathcal{F} \mid \mathcal{F}' = T(\mathcal{G} \mid \mathcal{G}')$*, then* $\mathcal{F} \mid \mathcal{F}'$ *and* $\mathcal{G} \mid \mathcal{G}'$ *are Holant-indistinguishable.*

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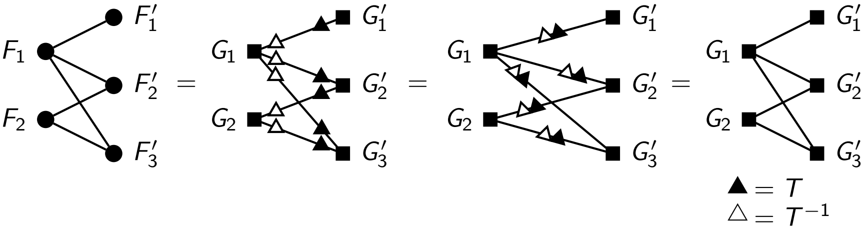

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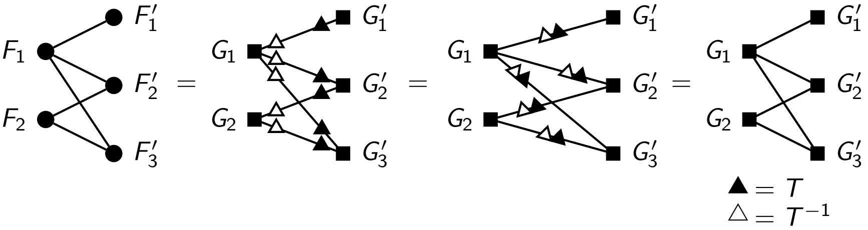

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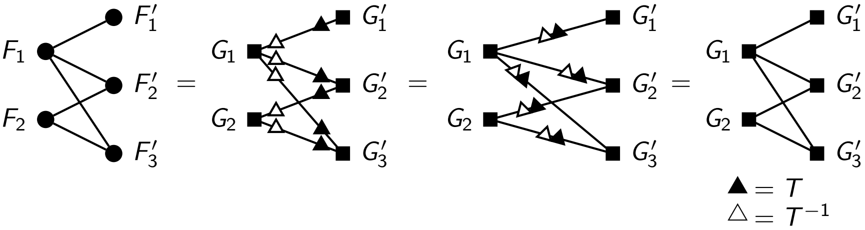

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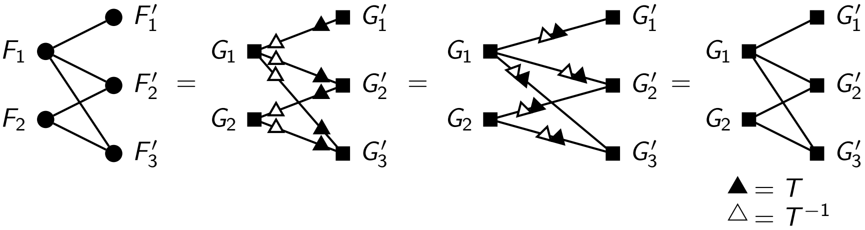

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- We prove two near-converses.

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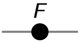
$\iff$ 

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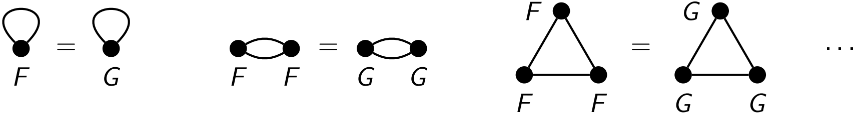
Bi-Holant-indistinguishable



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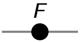
$\iff$ 

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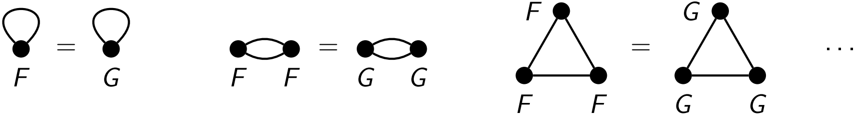

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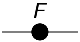
$\iff$ 

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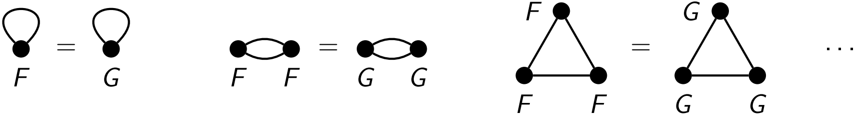

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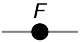
$\iff$ 

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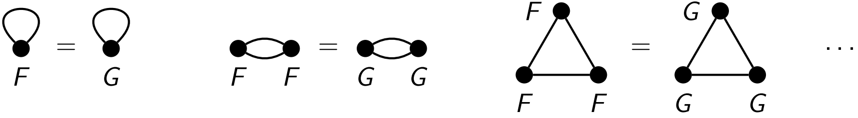

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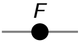
$\iff$ 

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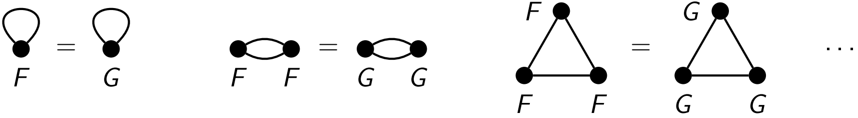

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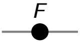
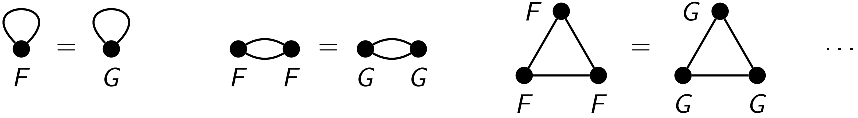
$\iff$  and  Bi-Holant-indistinguishable



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- Can't transpose F and G without $\supset$.

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- We generalize from matrices to arbitrary sets of tensors:

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- We generalize from matrices to arbitrary sets of tensors:

Theorem (The Approximate Converse [Cai-Y.'26])

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The Approximate Converse: Orbit Closure Intersection

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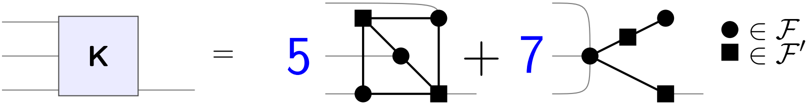
- Holant_Ω capture all GL_q-invariant polynomials!

- A **quantum gadget** is a formal linear combination of gadgets.



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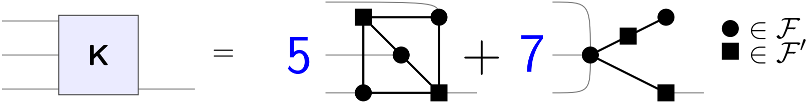
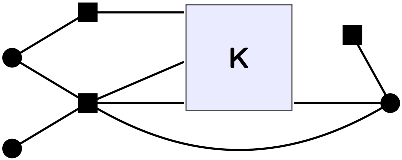


Definition

$\mathcal{F} \mid \mathcal{F}'$ is **quantum-nonvanishing** if $\forall$ quantum $\mathcal{F} \mid \mathcal{F}'$-gadget $\mathbf{K} \neq 0$,
...



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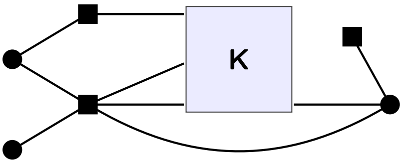
The Conditional Converse: Quantum Nonvanishing



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(non)Example

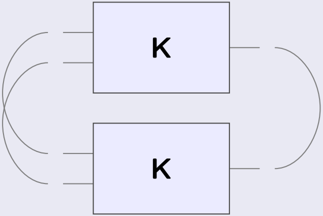
$\mathcal{F} \,|\, \mathcal{F}' = [\,1 \;\; i\,] \,|\, \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ is quantum-vanishing because every $\mathcal{F} \,|\, \mathcal{F}'$-grid has value 0:

$$[\,1 \;\; i\,] \; \bullet\!\!-\!\!-\!\!\bullet \; \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] = 1 \cdot 1 + i \cdot i = 0$$

The Conditional Converse: Quantum Nonvanishing

Example

$\mathbb{R}$-valued $\mathcal{F} \,|\, \mathcal{F}'$ is quantum-nonvanishing if $\subset \in \mathcal{F}$ and $\supset \in \mathcal{F}'$:



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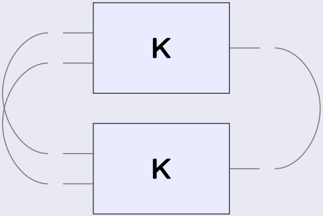

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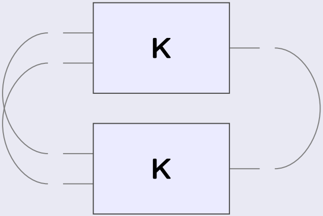

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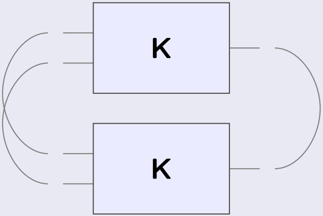

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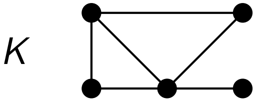
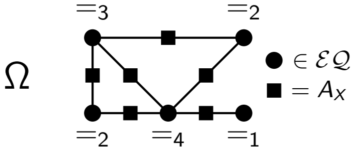
- Proof uses another version of Tannaka-Krein duality [Derksen-Makam'23].

Bounded-degree graph homomorphisms

- Recall: $\#\mathrm{hom}(\cdot, X) \equiv \mathrm{Holant}(\{A_X\} \mid \mathcal{EQ})$.





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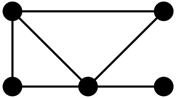

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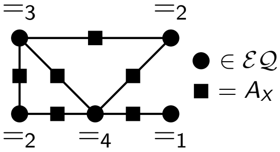

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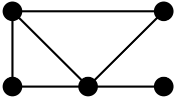

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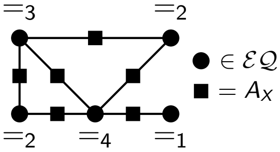

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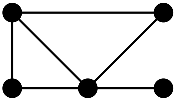

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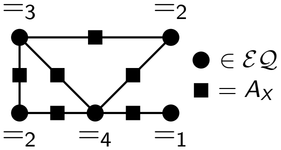

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Corollary ([Cai-Y.'26])

X and Y are homomorphism-indistinguishable over graphs of degree $\leq d$ iff $\overline{\mathrm{GL}_q(\{A_X\} \mid \mathcal{EQ}_{\leq d})}$ and $\overline{\mathrm{GL}_q(\{A_Y\} \mid \mathcal{EQ}_{\leq d})}$ intersect. This is decidable.

Thank you!

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