General and Planar #CSP Equality Corresponds to Classical and Quantum Isomorphism

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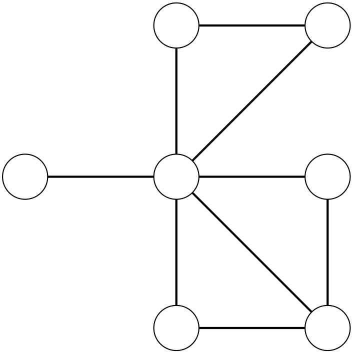
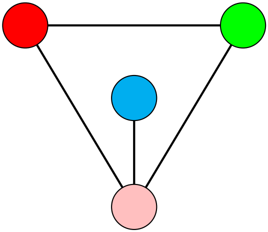
November 9, 2023

A mapping $\phi : V(K) \to V(G)$ is a **graph homomorphism** if it maps all edges to edges: $\{u, v\} \in E(K) \implies \{\phi(u), \phi(v)\} \in E(G)$.



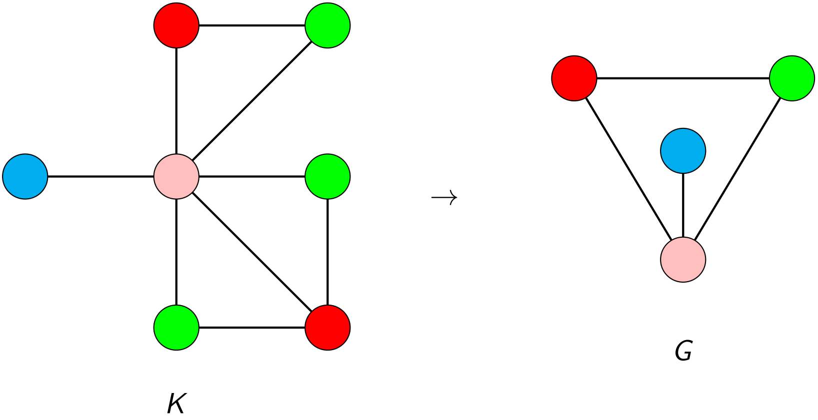
K



G

Graph Homomorphism

A mapping $\phi : V(K) \to V(G)$ is a **graph homomorphism** if it maps all edges to edges: $\{u, v\} \in E(K) \implies \{\phi(u), \phi(v)\} \in E(G)$.



Let $\hom(K, G)$ be the number of graph homomorphisms from K to G.

Theorem (Lovász'67)

$G \cong H$ *iff* $[\,\hom(K, G) = \hom(K, H)\ \forall\ \textit{graph}\ K]$.

Goal: Extend this theorem to #CSP.

The partition function for graph homomorphism

$$\begin{aligned}\hom(K,G) &= \sum_{\phi:V(K)\to V(G)} \mathbb{1}_{\phi \text{ is a homomorphism } K\to G} \\ &= \sum_{\phi:V(K)\to V(G)} \prod_{(u,v)\in E(K)} \mathbb{1}_{(\phi(u),\phi(v))\in E(G)} \\ &= \sum_{\phi:V(K)\to V(G)} \prod_{(u,v)\in E(K)} A_G(\phi(u),\phi(v))\end{aligned}$$

- A_G is adjacency matrix of G.

Counting graph homomorphisms is a **counting constraint satisfation problem** (#CSP).

- F is a **constraint function** on **domain** $D(F)$.
  - F of arity n is map $F : D(F)^n \to \mathbb{R}$
- #CSP$(\cdot, F)$ input is **instance** $\mathbf{K} = (X, C)$
  - X is set of **variables**
  - $(x_{i_1}, \ldots, x_{i_n}) \in C$ is a **constraint** applying F to variables $x_{i_1}, \ldots, x_{i_n}$.

$$\#\mathrm{CSP}((X, C), F) = \sum_{\phi: X \to D(F)} \prod_{(x_{i_1}, \ldots, x_{i_n}) \in C} F(\phi(x_{i_1}), \ldots, \phi(x_{i_n})).$$

$\hom(K, G)$ has variable set $X = V(K)$, domain $D(F) = V(G)$, constraint set $C = E(K)$, and constraint function $F = A_G$.

$$\hom(K, G) = \sum_{\phi: V(K) \to V(G)} \prod_{(u,v) \in E(K)} A_G(\phi(u), \phi(v))$$
$$= \#\mathrm{CSP}\big(\big(V(K), E(K)\big), A_G\big).$$

Why study #CSP?

- Beautiful complexity dichotomy theorems

Theorem (Cai-Chen'17)

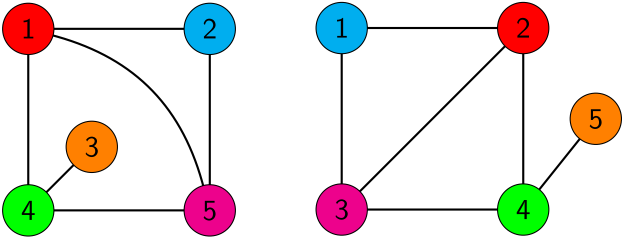
For any finite set $\mathcal{F}$ of $\mathbb{C}$-valued constraint functions, $\#$CSP$(\cdot, \mathcal{F})$ is always either in P or $\#$P-hard, with nothing in between.

Theorem (Cai-Fu'19)

For any set $\mathcal{F}$ of $\mathbb{C}$-valued constraint functions over Boolean variables, $\#$CSP$(\cdot, \mathcal{F})$ is exactly one of the following:

1. *P-time solvable;*
2. *P-time solvable over planar graphs but $\#$P-hard over general graphs;*
3. *$\#$P-hard over planar graphs.*

Constraint function isomorphism



$\sigma(1) = 2$
$\sigma(2) = 1$
$\sigma(4) = 4$
$\sigma(5) = 3$
$\sigma(3) = 5$

- Graphs $G \cong H$ if $\exists$ a bijection $\sigma : V(G) \to V(H)$ such that

$$A_G(u, v) = A_H(\sigma(u), \sigma(v))$$

  for every u, v.
- A_G, A_H are binary constraint functions.
- n-ary $F \cong G$ if $\exists$ a bijection $\sigma : D(F) \to D(G)$ such that

$$F(x_1, \ldots, x_n) = G(\sigma(x_1), \ldots, \sigma(x_n))$$

  for every $x_1, \ldots, x_n$.

#CSP invariance determines isomorphism

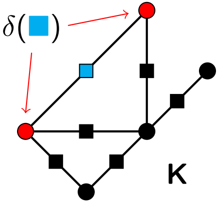
Theorem (Lovász'67)

$G \cong H$ *iff* $[\ \hom(K, G) = \hom(K, H)\ \forall$ *graph* $K]$.

Theorem (Y. '23)

$F \cong G$ *iff* $[\ \#\mathrm{CSP}(\mathbf{K}, F) = \#\mathrm{CSP}(\mathbf{K}, G)\ \forall$ *#CSP instance* $\mathbf{K}]$.

- $A_G : V(G)^2 \to \{0, 1\}$ is
  - Binary
  - Symmetric
  - 0-1-valued
- Our F is in general
  - Arbitrary-arity,
  - Asymmetric,
  - Real-valued.

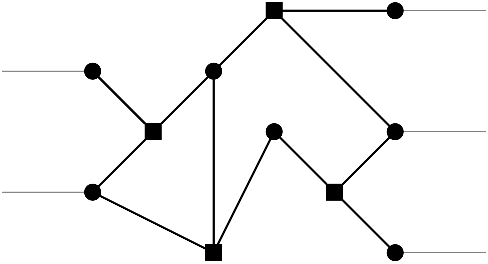


- Constraint-variable incidence graph
- $\bullet$ – variable
- $\blacksquare$ – constraint (F)
- $\blacksquare$ applied to adjacent variables $\delta(\blacksquare)$.

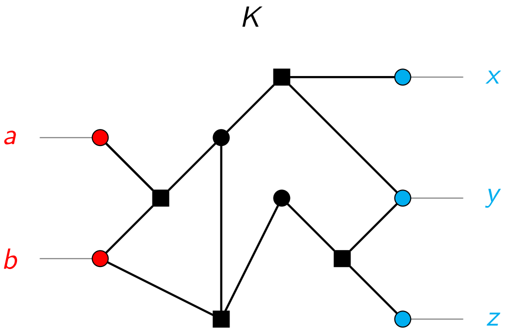
$$\#\mathrm{CSP}(\mathbf{K}, F) = \sum_{\phi:\{\bullet\}\to D(F)} \prod_{\blacksquare} F(\phi(\delta(\blacksquare))).$$

Gadgets

- A **gadget** is a signature grid with **dangling edges**.
- Several constraint functions assembled into a new constraint.
- Inputs along dangling edges.



Gadgets and signature matrices

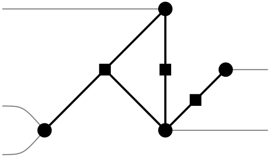


- $a, b, x, y, z \in D(F)$
- $|D(F)|^2 \times |D(F)|^3$ signature matrix $M(\mathbf{K})$.

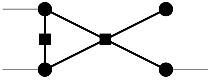
$$M(\mathbf{K})_{ab,xyz} = \sum_{\substack{\phi:\{\bullet\}\to D(F) \\ \phi(\bullet,\bullet)=(a,b) \\ \phi(\bullet,\bullet,\bullet)=(x,y,z)}} \prod_{\blacksquare} F(\phi(\delta(\blacksquare))).$$

Gadget operations

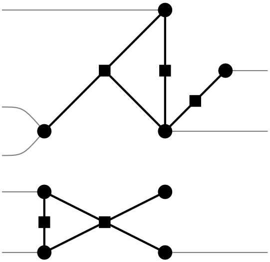
K



L

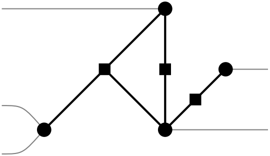


K $\otimes$ L

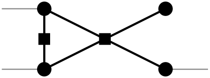


Gadget operations

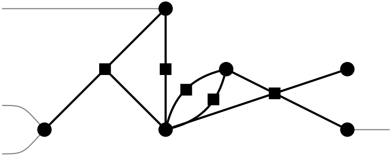
K



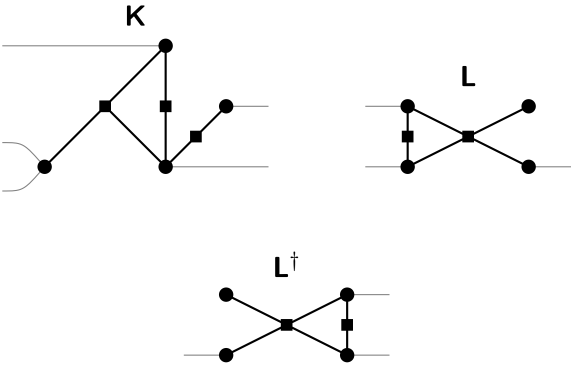
L



K ∘ L



Gadget operations



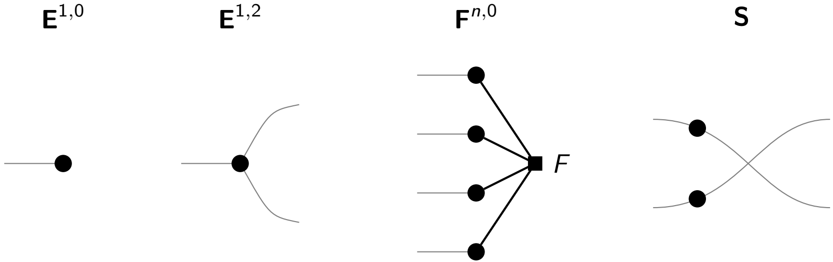
Gadgets operations

- Gadget operations correspond to signature matrix operations:
  - $M(\mathbf{K} \otimes \mathbf{L}) = M(\mathbf{K}) \otimes M(\mathbf{L})$
  - $M(\mathbf{K} \circ \mathbf{L}) = M(\mathbf{K})M(\mathbf{L})$
  - $M(\mathbf{K}^\dagger) = M(\mathbf{K})^\dagger$

Building block gadgets

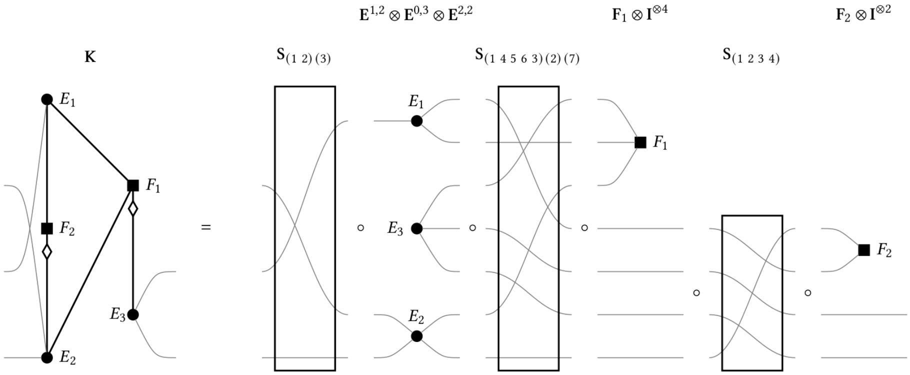
Theorem

The set of all $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0}, \mathbf{S} \rangle_{\circ,\otimes,\dagger}$.



$$M(\mathbf{E}^{1,2})_{x,yz} = \begin{cases} 1 & x = y = z \\ 0 & \text{otherwise} \end{cases}.$$

Gadget decomposition



Constraint function isomorphism revisited

Recall: n-ary $F \cong G$ if $\exists$ a bijection $\sigma : D(F) \to D(G)$ such that

$$F(x_1, \dots, x_n) = G(\sigma(x_1), \dots, \sigma(x_n))$$

for every $x_1, \dots, x_n$.

- Let P_σ be the permutation matrix for σ
- Let $f, g \in \mathbb{R}^{|D(F)|^n}$ be vectorizations of F, G
- $P_\sigma^{\otimes n} f = g$
- Think: apply σ to each axis
- $F \cong G$ iff there is a permutation matrix P s.t. $P^{\otimes n} f = g$.
- $\mathrm{Aut}(F) = \{P : P^{\otimes n} f = f\}$.

The intertwiner space

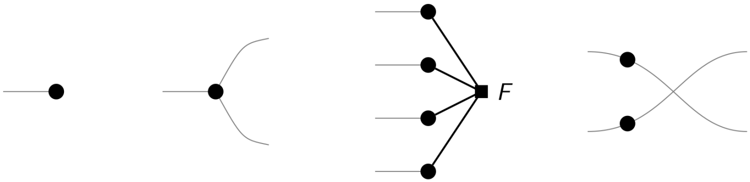
$$C_{\mathrm{Aut}(F)} = \left\{ \text{ matrix } A \;\middle|\; P^{\otimes m} A = A P^{\otimes d} \; \forall P \in \mathrm{Aut}(F) \right\}$$

All the matrices A fixed by every automorphism of F

Lemma

$$C_{\mathrm{Aut}(F)} = \mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}), M(\mathbf{S})\rangle_{\circ,\otimes,\dagger}).$$

$\mathbf{E}^{1,0}$ $\mathbf{E}^{1,2}$ $\mathbf{F}^{n,0}$ $\mathbf{S}$



- $M(\mathbf{F}^{n,0}) = f \in C_{\mathrm{Aut}(F)}$ (because $P^{\otimes n} f = f$ if $P \in \mathrm{Aut}(F)$).
- $M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{S})$ fixed under all permutations, so in $C_{\mathrm{Aut}(F)}$.
- $\subseteq$: *Tannaka-Krein duality* for classical permutation groups.

Intertwiners to Isomorphism

Lemma

$$C_{\mathrm{Aut}(F)} = \mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}), M(\mathbf{S})\rangle_{\circ,\otimes,\dagger}).$$

Recall:

Theorem

The set of all $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0}, \mathbf{S}\rangle_{\circ,\otimes,\dagger}$.

Corollary

$$C_{\mathrm{Aut}(F)} = \mathrm{span}(\text{Signature matrices of } \#\mathrm{CSP}(\cdot, F) \text{ gadgets}).$$

Intertwiners to Isomorphism

Corollary

$$C_{\mathrm{Aut}(F)} = \text{span}(\text{Signature matrices of } \#\text{CSP}(\cdot, F) \text{ gadgets}).$$

Connection between $\mathrm{Aut}(F)$ and gadgets gives:

Lemma

Let $x, y \in D(F)$. If $M(\mathbf{K})_x = M(\mathbf{K})_y$ for every gadget $\mathbf{K}$ with one dangling edge, then $\exists \sigma \in \mathrm{Aut}(F)$ s.t. $\sigma(x) = y$.

- Create F': add a domain elt v_f to $D(F)$ 'adjacent' to every other domain elt.
- Create G' similarly by adding v_g.
- Apply Lemma to v_f and v_g and 'disjoint union' $F' \sqcup G'$
- (use assumption that $\#\text{CSP}(\mathbf{K}, F) = \#\text{CSP}(\mathbf{K}, G)$ for every $\mathbf{K}$).
- Gives $\sigma \in \mathrm{Aut}(F' \sqcup G')$ such that $\sigma(v_f) = v_g$
- F' and G' are 'connected', so $F \cong G$.

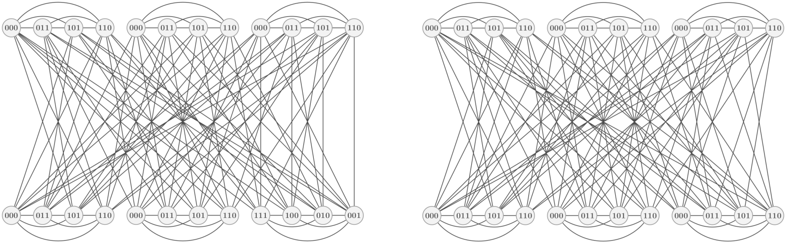
Quantum Isomorphism

- Recall $F \cong G$ if there is a permutation matrix P such that $P^{\otimes n} f = g$.
- Say $F \cong_q G$ (F and G are **quantum isomorphic**) if there is a **quantum permutation matrix** $\mathcal{U}$ such that $\mathcal{U}^{\otimes n} f = g$.
- Quantum isomorphism originally defined in terms of a quantum nonlocal game [AMRSSV '18].
  - Defined a cooperative nonlocal game s.t. players have win probability 1 iff two graphs are isomorphic
  - Graphs are quantum isomorphic iff the players have win probability 1 when allowed to measure a shared quantum state.

Quantum Isomorphism

There are pairs of graphs which are quantum isomorphic but not isomorphic!

- From theory of quantum nonlocal games [AMRSSV '18]



- Non-isomorphic Hadamard graphs [Chan and Martin '24]

A **quantum permutation matrix** is an abstract relaxation of a permutation matrix.

- Entries come from an abstract space instead of $\{0, 1\}$.
- Entries don't necessarily commute.
- Rows and columns still add up to 1
- Product of distinct elements in same row or column in 0
- Any quantum permutation matrix whose entries commute is a permutation matrix.

Quantum Isomorphism and planarity

Theorem (Lovász'67)

$G \cong H$ *iff* $[\ \hom(K,G) = \hom(K,H)\ \forall$ *graph* $K]$.

Theorem (Mančinska-Roberson '19)

$G \cong_q H$ *iff* $[\ \hom(K,G) = \hom(K,H)\ \forall$ *planar graph* $K]$.

Theorem (Y.'23)

$F \cong G$ *iff* $[\ \#\mathrm{CSP}(\mathbf{K},F) = \#\mathrm{CSP}(\mathbf{K},G)\ \forall$ *#CSP instance* $\mathbf{K}]$.

Theorem (Cai-Y.'23)

$F \cong_q G$ *iff* $[\ \#\mathrm{CSP}(\mathbf{K},F) = \#\mathrm{CSP}(\mathbf{K},G)\ \forall$ *planar #CSP instance* $\mathbf{K}]$.

- #CSP instance is planar if its signature grid (constraint-variable incidence graph) is planar.
- What does planarity have to do with noncommutativity?

The planar gadget decomposition

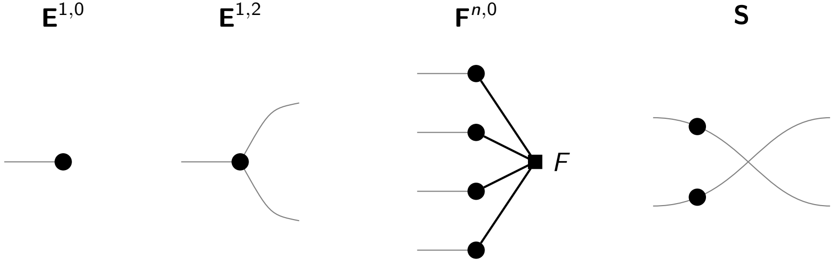
Recall:

Theorem

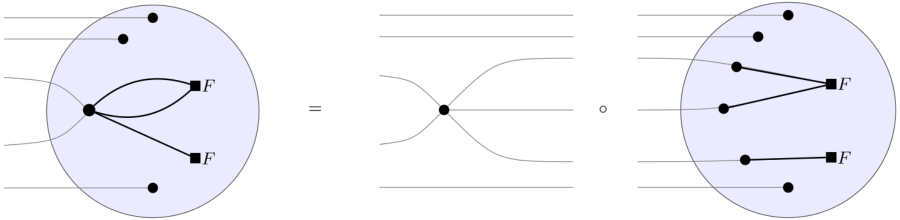
The set of all $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0}, \mathbf{S} \rangle_{\circ,\otimes,\dagger}$.

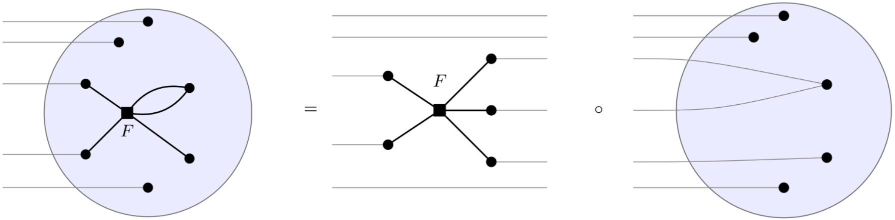
Theorem

The set of all planar $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0} \rangle_{\circ,\otimes,\dagger}$.



- Can decompose any planar gadget into a chain of simple gadgets





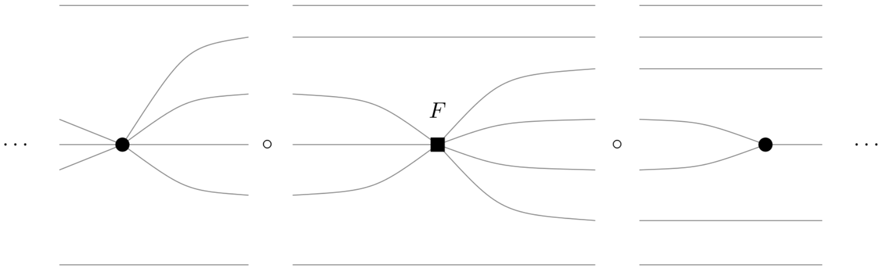
Recall our main theorem:

Theorem (Cai-Y.'23)

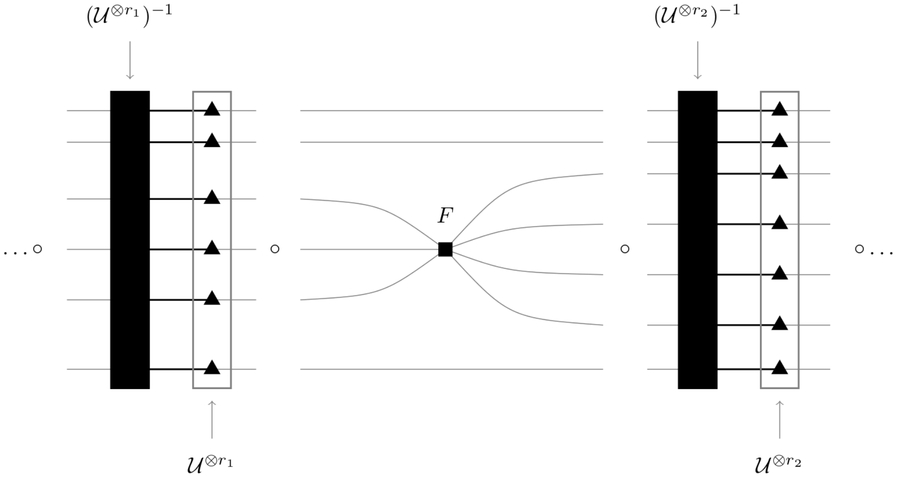
$F \cong_{qc} G$ *iff* $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)$ $\forall$ *planar* $\mathbf{K}$.

- Suppose $F \cong_{qc} G$, so $\mathcal{U}^{\otimes n} f = g$ for quantum permutation matrix $\mathcal{U}$.
- ($\Longrightarrow$): View $\mathcal{U}$ itself as a constraint function.

A quantum holographic transformation

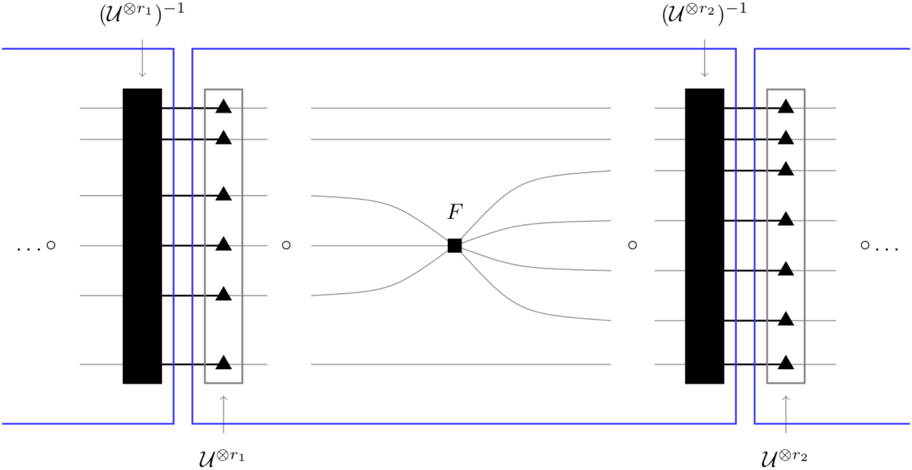


The planar gadget decomposition converts $\mathbf{K}$ to a composition of building block gadgets.



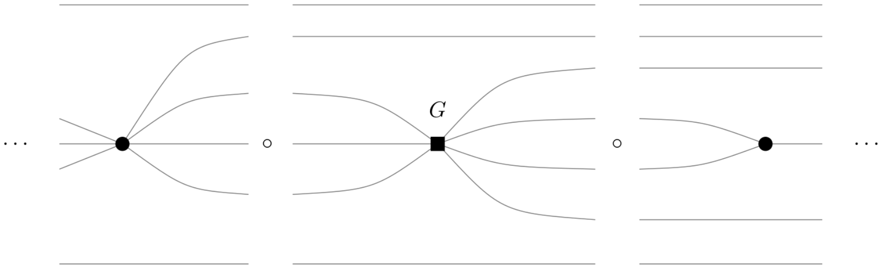
Insert $(\mathcal{U}^{\otimes r_i})^{-1}\mathcal{U}^{\otimes r_i} = I$ between the ith and $(i+1)$st factors (preserves $\#\mathrm{CSP}(\mathbf{K}, F)$ value).

A quantum holographic transformation



Reassociate. Now, $\mathcal{U}^{\otimes n} f = g$ and $\mathcal{U}$ doesn't affect • vertices, so...

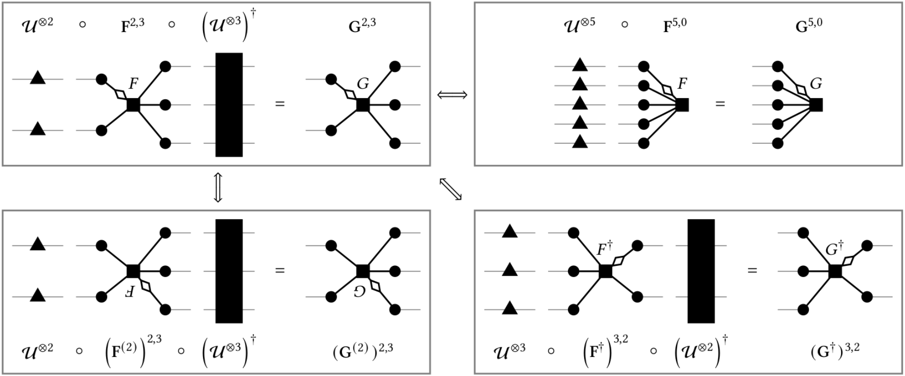
A quantum holographic transformation



Every F is converted to G without changing the #CSP value: $\#\text{CSP}(\mathbf{K}, F) = \#\text{CSP}(\mathbf{K}, G)$.

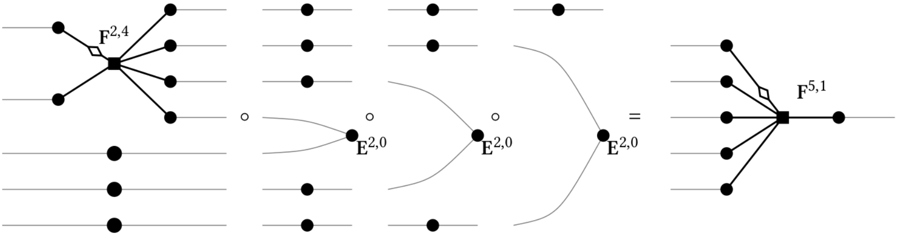
A quantum holographic transformation

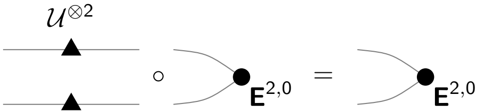
- Can't view $\mathcal{U}$ as a constraint function in general (nonplanar) signature grids because entries of $\mathcal{U}$ don't commute.
    - #CSP value is a sum of products of constraint function evaluations.
- Planar gadget decomposition gives order of vertices
    - hence multiplication order of $\mathcal{U}$ entries.

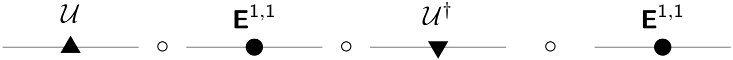


$\mathcal{U}^{\otimes 2} \circ \mathbf{F}^{2,3} \circ \left(\mathcal{U}^{\otimes 3}\right)^{\dagger}$ $\mathbf{G}^{2,3}$
F
G
=
$\mathcal{U}^{\otimes 5} \circ \mathbf{F}^{5,0}$ $\mathbf{G}^{5,0}$
F
G
=
=
=
$F^{\dagger}$
$G^{\dagger}$
$\mathcal{U}^{\otimes 2} \circ \left(\mathbf{F}^{(2)}\right)^{2,3} \circ \left(\mathcal{U}^{\otimes 3}\right)^{\dagger}$ $(\mathbf{G}^{(2)})^{2,3}$
$\mathcal{U}^{\otimes 3} \circ \left(\mathbf{F}^{\dagger}\right)^{3,2} \circ \left(\mathcal{U}^{\otimes 2}\right)^{\dagger}$ $(\mathbf{G}^{\dagger})^{3,2}$

Planar Symmetries







Theorem (Cai-Y.'23)

$F \cong_{qc} G$ *iff* $\#CSP(\mathbf{K}, F) = \#CSP(\mathbf{K}, G)$ $\forall$ *planar* $\mathbf{K}$.

- Next, prove ($\Leftarrow$). Similar techniques to classical/nonplanar proof, but more involved.

Definition

Quantum permutation matrix $\mathcal{U}$ s.t. $\mathcal{U}^{\otimes n} f = f$ defines the *quantum automorphism group* $\mathrm{Qut}(F)$ of F.

- Recall $\mathcal{U}^{\otimes n} f = g$ defined quantum isomorphism of F and G.
- Instead of studying $\mathrm{Qut}(F)$ directly, study its intertwiner space

$$C_{\mathrm{Qut}(F)} = \left\{ \text{matrix } A \;\middle|\; \mathcal{U}^{\otimes m} A = A \mathcal{U}^{\otimes d} \right\}$$

Recall:

Theorem

The set of all $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0}, \mathbf{S} \rangle_{\circ,\otimes,\dagger}$.

Lemma

$$\begin{aligned} C_{\mathrm{Aut}(F)} &= \mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}), M(\mathbf{S}) \rangle_{\circ,\otimes,\dagger}) \\ &= \mathrm{span}(\text{Signature matrices of } \#\mathrm{CSP}(\cdot, F) \text{ gadgets}) \end{aligned}$$

Theorem

The set of all planar $\#\mathrm{CSP}(\cdot, F)$ gadgets is exactly $\langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0} \rangle_{\circ,\otimes,\dagger}$.

Now this follows similarly to classical case using Tannaka-Krein duality:

Lemma

$$\begin{aligned} C_{\mathrm{Qut}(F)} &= \mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}) \rangle_{\circ,\otimes,\dagger}) \\ &= \mathrm{span}(\text{Signature matrices of planar } \#\mathrm{CSP}(\cdot, F) \text{ gadgets}). \end{aligned}$$

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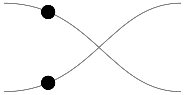
S



- **S** allows for nonplanar gadgets.
- Also, $\mathbf{S} \in C_{\mathrm{Qut}(F)}$ iff entries of $\mathcal{U}$ defining $\mathrm{Qut}(F)$ commute!

The converse

Theorem (Cai-Y.'23)

$F \cong_q G$ *iff* [$\#\mathrm{CSP}(\mathbf{K}, F) = \hom(\mathbf{K}, G)$ $\forall$ *planar #CSP instance* $\mathbf{K}$].

Lemma

$$\begin{aligned} \mathcal{C}_{\mathrm{Qut}(F)} &= \mathrm{span}(\langle M(\mathbf{E}^{1,0}), M(\mathbf{E}^{1,2}), M(\mathbf{F}^{n,0}), M(\mathbf{S})\rangle_{\circ,\otimes,\dagger}) \\ &= \mathrm{span}(\text{Signature matrices of planar } \#\mathrm{CSP}(\cdot, F) \text{ gadgets}) \end{aligned}$$

Lemma

Let $x, y \in D(F)$. If $M(\mathbf{K})_x = M(\mathbf{K})_y$ for every planar gadget $\mathbf{K}$ with one dangling edge, then F has a 'quantum automorphism' mapping x to y.

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- Trick from classical case still works for quantum isomorphism:
  - Requires theory of orbits of quantum permutation groups [Lupini, Mančinska and Roberson '17]
- Add vertices v_f to F and v_g to G adjacent to all other vertices
- Apply Lemma to v_f and v_g and 'disjoint union' $F' \sqcup G'$.
- Gives quantum automorphism of $F' \sqcup G'$ sending v_f to v_g.
- $F \cong_q G$.

Thank you!
Questions?